6.4 POLAR COORDINATES

What you'll learn about

- Polar Coordinate System
- Coordinate Conversion
- Equation Conversion
- Finding Distance Using Polar Coordinates

. . . and why

Use of polar coordinates sometimes simplifies complicated rectangular equations and they are useful in calculus.

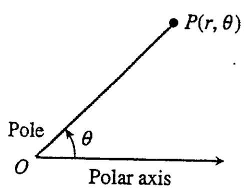


FIGURE 6.35 The polar coordinate system.

Polar Coordinate System

A **polar coordinate system** is a plane with a point O, the **pole**, and a ray from O, the **polar axis**, as shown in Figure 6.35. Each point P in the plane is assigned **polar coordinates** as follows: r is the **directed distance** from O to P, and θ is the **directed angle** whose initial side is on the polar axis and whose terminal side is on the line OP.

As in trigonometry, we measure θ as positive when moving counterclockwise and negative when moving clockwise. If $r > 0$, then P is on the terminal side of θ. If $r < 0$, then P is on the terminal side of $\theta + \pi$. We can use radian or degree measure for the angle θ as illustrated in Example 1.

EXAMPLE 1 Plotting points in the polar coordinate system

Plot the points with the given polar coordinates.

(a) $P(2, \pi/3)$ **(b)** $Q(-1, 3\pi/4)$ **(c)** $R(3, -45°)$

SOLUTION Figure 6.36 shows the three points. Now try Exercise 7.

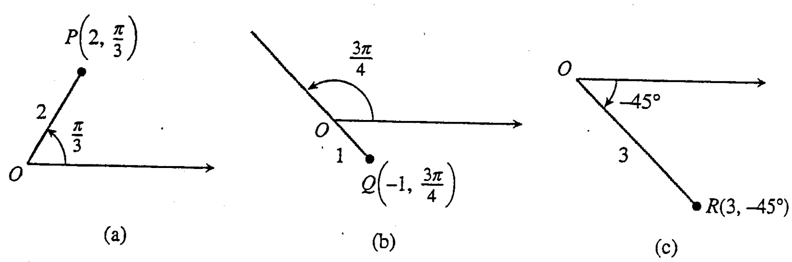


FIGURE 6.36 The three points in Example 1.

Each polar coordinate pair determines a unique point. However, the polar coordinates of a point P in the plane are not unique.

EXAMPLE 2 Finding all polar coordinates for a point

If the point P has polar coordinates $(3, \pi/3)$, find all polar coordinates for P.

SOLUTION Point P is shown in Figure 6.37. Two additional pairs of polar coordinates for P are

$$\left(3, \frac{\pi}{3} + 2\pi\right) = \left(3, \frac{7\pi}{3}\right) \quad \text{and} \quad \left(-3, \frac{\pi}{3} + \pi\right) = \left(-3, \frac{4\pi}{3}\right).$$

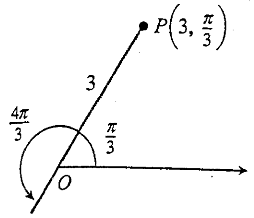


FIGURE 6.37 The point P in Example 2.

We can use these two pairs of polar coordinates for P to write the rest of the possibilities:

$$\left(3, \frac{\pi}{3} + 2n\pi\right) = \left(3, \frac{(6n+1)\pi}{3}\right) \text{ or}$$

$$\left(-3, \frac{\pi}{3} + (2n+1)\pi\right) = \left(-3, \frac{(6n+4)\pi}{3}\right)$$

Where n is any integer. Now try Exercise 23.

The coordinates (r, θ), $(r, \theta + 2\pi)$, and $(-r, \theta + \pi)$ all name the same point. In general, the point with polar coordinates (r, θ) also has the following polar coordinates:

Finding all Polar Coordinates of a Point

Let P have polar coordinates (r, θ). Any other polar coordinate of P must be of the form

$$(r, \theta + 2n\pi) \text{ or } (-r, \theta + (2n+1)\pi)$$

where n is any integer. In particular, the pole has polar coordinates $(0, \theta)$, where θ is any angle.

Coordinate Conversion

When we use both polar coordinates and Cartesian coordinates, the pole is the origin and the polar axis is the positive x-axis as shown in Figure 6.38. By applying trigonometry we can find equations that relate the polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P.

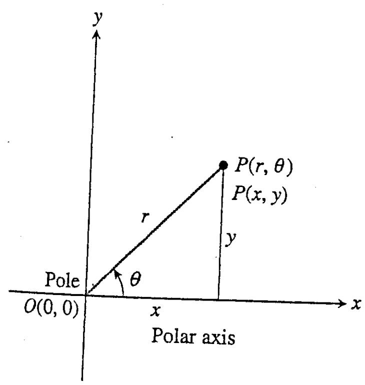


FIGURE 6.38 Polar and rectangular coordinates for P.

Coordinate Conversion Equations

Let the point P have polar coordinates (r, θ) and rectangular coordinates (x, y). Then

$$x = r\cos\theta, \quad r^2 = x^2 + y^2,$$

$$y = r\sin\theta, \quad \tan\theta = \frac{y}{x}$$

These relationships allow us to convert from one coordinate system to the other.

EXAMPLE 3 Converting from polar to rectangular coordinates

Find the rectangular coordinates of the points with the given polar coordinates.

(a) $P(3, 5\pi/6)$ **(b)** $Q(2, -200°)$

SOLUTION

(a) For $P(3, 5\pi/6)$, $r = 3$ and $\theta = 5\pi/6$:

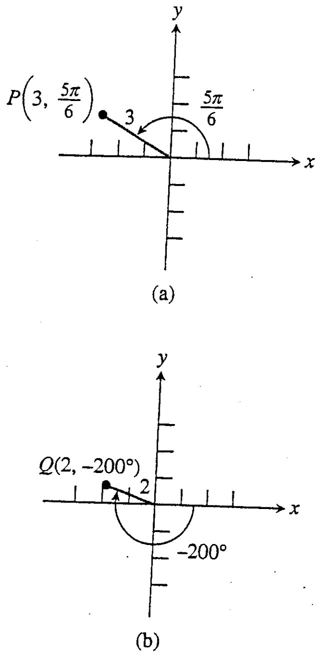


FIGURE 6.39 The points P and Q in Example 3.

$$x = r \cos \theta \qquad \qquad y = r \sin \theta$$

$$x = 3 \cos \frac{5\pi}{6} \qquad \text{and} \qquad y = 3 \sin \frac{5\pi}{6}$$

$$x = 3\left(-\frac{\sqrt{3}}{2}\right) \approx -2.60 \qquad \qquad y = 3\left(\frac{1}{2}\right) = 1.5$$

The rectangular coordinates for P are $(-3\sqrt{3}/2,\ 1.5) \approx (-2.60,\ 1.5)$ (Figure 6.39a).

(b) For $Q(2, -200°)$, $r = 2$ and $\theta = -200°$:

$$x = r \cos \theta \qquad \qquad y = r \sin \theta$$

$$x = 2 \cos(-200°) \approx -1.88 \qquad \text{and} \qquad y = 2 \sin(-200°) \approx 0.68$$

The rectangular coordinates for Q are approximately $(-1.88,\ 0.68)$ (Figure 6.39b).

Now try Exercise 15.

When converting rectangular coordinates to polar coordinates, we must remember that there are infinitely many possible polar coordinate pairs. In Example 4 we report two of the possibilities.

EXAMPLE 4 Converting from rectangular to polar coordinates

Find two polar coordinate pairs for the points with given rectangular coordinates.

(a) $P(-1, 1)$ **(b)** $Q(-3, 0)$

SOLUTION

(a) For $P(-1, 1)$, $x = -1$ and $y = 1$:

$$r^2 = x^2 + y^2 \qquad \qquad \tan \theta = \frac{y}{x}$$

$$r^2 = (-1)^2 + (1)^2 \qquad \qquad \tan \theta = \frac{-1}{1} = -1$$

$$r = \pm\sqrt{2} \qquad \qquad \theta = \tan^{-1}(-1) + n\pi = -\frac{\pi}{4} + n\pi$$

We use the angles $-\pi/4$ and $-\pi/4 + \pi = 3\pi/4$. Because P is on the ray opposite the terminal side of $-\pi/4$, the value of r corresponding to this angle is negative (Figure 6.40). Because P is on the terminal side of $3\pi/4$, the value of r corresponding to this angle is positive. So two polar coordinate pairs of point P are

$$\left(-\sqrt{2}, -\frac{\pi}{4}\right) \quad \text{and} \quad \left(\sqrt{2}, \frac{3\pi}{4}\right).$$

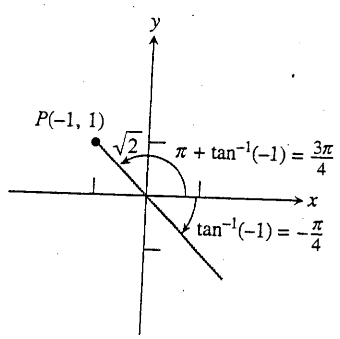


FIGURE 6.40 The point P in Example 4a.

(b) For $Q(-3, 0)$, $x = -3$ and $y = 0$. Thus, $r = \pm 3$ and $\theta = n\pi$. We use the angles 0 and π. So two polar coordinates pairs for point Q are

$$(-3, 0) \text{ and } (3, \pi).$$

Now try Exercise 27.

QUICK REVIEW 6.4

(For help, go to Sections P.2, 4.3, and 5.6.)

In Exercises 1 and 2, determine the quadrants containing the terminal side of the angles.

1. (a) $5\pi/6$ **(b)** $-3\pi/4$

2. (a) $-300°$ **(b)** $210°$

In Exercises 3–6, find a positive and a negative angle coterminal with the given angle.

3. $-\pi/4$

4. $\pi/3$

5. $160°$

6. $-120°$

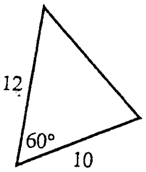
In Exercises 7 and 8, write a standard form equation for the circle.

7. Center (3, 0) and radius 2

8. Center (0, −4) and radius 3

In Exercises 9 and 10, use The Law of Cosines to find the measure of the third side of the given triangle.

9.



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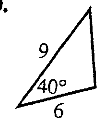
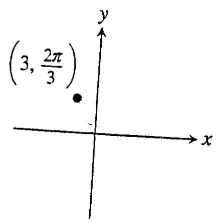


SECTION 6.4 EXERCISES

In Exercises 1–4, the polar coordinates of a point are given. Find its rectangular coordinates.

1.



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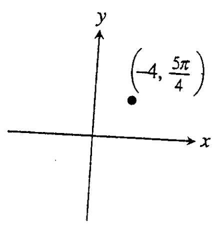

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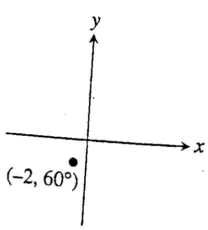

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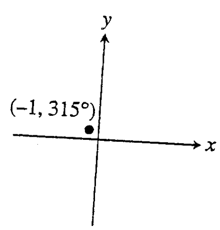


In Exercises 5 and 6, **(a)** complete the table for the polar equation and **(b)** plot the corresponding points.

5. $r = 3 \sin \theta$

θ	$\pi/4$	$\pi/2$	$5\pi/6$	π	$4\pi/3$	2π
r						

6. $r = 2 \csc \theta$

θ	$\pi/4$	$\pi/2$	$5\pi/6$	π	$4\pi/3$	2π
r						

In Exercises 7–14, plot the point with the given polar coordinates.

7. $(3, 4\pi/3)$

8. $(2, 5\pi/6)$

9. $(-1, 2\pi/5)$

10. $(-3, 17\pi/10)$

11. $(2, 30°)$

12. $(3, 210°)$

13. $(-2, 120°)$

14. $(-3, 135°)$

In Exercises 15–22, find the rectangular coordinates of the point with given polar coordinates.

15. $(1.5, 7\pi/3)$

16. $(2.5, 17\pi/4)$

17. $(-3, -29\pi/7)$

18. $(-2, -14\pi/5)$

19. $(-2, \pi)$

20. $(1, \pi/2)$

21. $(2, 270°)$

22. $(-3, 360°)$

In Exercises 23–26, polar coordinates of point P are given. Find all of its polar coordinates.

23. $P = (2, \pi/6)$

24. $P = (1, -\pi/4)$

25. $P = (1.5, -20°)$

26. $P = (-2.5, 50°)$

In Exercises 27–30, rectangular coordinates of point P are given. Find all polar coordinates of P that satisfy

(a) $0 \le \theta \le 2\pi$ **(b)** $-\pi \le \theta \le \pi$ **(c)** $0 \le \theta \le 4\pi$

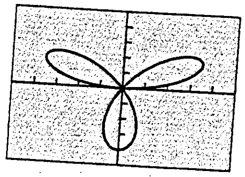
27. $P = (1, 1)$

28. $P = (1, 3)$

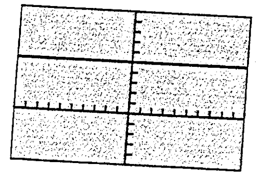
29. $P = (-2, 5)$

30. $P = (-1, -2)$

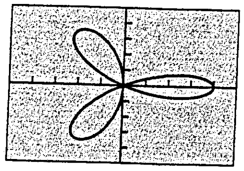
In Exercises 31–34, use your grapher to match the polar equation with its graph.



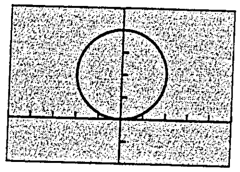
(a)



(b)



(c)



(d)

31. $r = 5 \csc \theta$

32. $r = 4 \sin \theta$

33. $r = 4 \cos 3\theta$

34. $r = 4 \sin 3\theta$

In Exercises 35–42, convert the polar equation to rectangular form and identify the graph. Support your answer by graphing the polar equation.

35. $r = 3 \sec \theta$ **36.** $r = -2 \csc \theta$ **37.** $r = -3 \sin \theta$

38. $r = -4 \cos \theta$ **39.** $r \csc \theta = 1$ **40.** $r \sec \theta = 3$

41. $r = 2 \sin \theta - 4 \cos \theta$ **42.** $r = 4 \cos \theta - 4 \sin \theta$

In Exercises 43–50, convert the rectangular equation to polar form. Graph the polar equation.

43. $x = 2$

44. $x = 5$

45. $2x - 3y = 5$

46. $3x + 4y = 2$

47. $(x - 3)^2 + y^2 = 9$

48. $x^2 + (y - 1)^2 = 1$

49. $(x + 3)^2 + (y + 3)^2 = 18$

50. $(x - 1)^2 + (y + 4)^2 = 17$

51. Tracking Airplanes The location, given in polar coordinates, of two planes approaching the Vicksburg airport are (4 mi, 12°) and (2 mi, 72°). Find the distance between the airplanes.

52. Tracking Ships The location of two ships from Mays Landing Lighthouse, given in polar coordinates, are (3 mi, 170°) and (5 mi, 150°). Find the distance between the ships.

53. Using Polar Coordinates in Geometry A square with sides of length a and center at the origin has two sides parallel to the x-axis. Find polar coordinates of the vertices.

54. Using Polar Coordinates in Geometry A regular pentagon whose center is at the origin has one vertex on the positive x-axis at a distance a from the center. Find polar coordinates of the vertices.

Standardized Test Questions

55. True or False Every point in the plane has exactly two polar coordinates. Justify your answer.

56. True or False If r_1 and r_2 are not 0, and if (r_1, θ) and $(r_2, \theta + \pi)$ represent the same point in the plane, then $r_1 = -r_2$. Justify your answer.

In Exercises 57–60, solve the problem without using a calculator.

57. Multiple Choice If $r \neq 0$, which of the following polar coordinate pairs represents the same point as the point with polar coordinates (r, θ)?

(a) $(-r, \theta)$ **(b)** $(-r, \theta + 2\pi)$ **(c)** $(-r, \theta + 3\pi)$

(d) $(r, \theta + \pi)$ **(e)** $(r, \theta + 3\pi)$

58. Multiple Choice Which of the following are the rectangular coordinates of the point with polar coordinate $(-2, -\pi/3)$?

(a) $(-\sqrt{3}, 1)$ **(b)** $(-1, -\sqrt{3})$ **(c)** $(-1, \sqrt{3})$

(d) $(1, -\sqrt{3})$ **(e)** $(1, \sqrt{3})$

59. Multiple Choice Which of the following polar coordinate pairs represent the same point as the point with polar coordinates (2, 110°)?

(a) (−2, −70°) **(b)** (−2, 110°) **(c)** (−2, −250°)

(d) (2, −70°) **(e)** (2, 290°)

60. Multiple Choice Which of the following polar coordinate pairs does *not* represent the point with rectangular coordinates (−2, −2)?

(a) $(2\sqrt{2}, -135°)$ **(b)** $(2\sqrt{2}, 225°)$

(c) $(-2\sqrt{2}, -315°)$ **(d)** $(-2\sqrt{2}, 45°)$

(e) $(-2\sqrt{2}, 135°)$

Explorations

61. Polar Distance Formula Let P_1 and P_2 have polar coordinates (r_1, θ_1) and (r_2, θ_2), respectively.

(a) If $\theta_1 - \theta_2$ is a multiple of π, write a formula for the distance between P_1 and P_2.

(b) Use the Law of Cosines to prove that the distance between P_1 and P_2 is given by

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}$$

(c) Writing to Learn Does the formula in (b) agree with the formula(s) you found in (a)? Explain.

62. Watching Your θ-Step Consider the polar curve $r = 4 \sin \theta$. Describe the graph for each of the following.

(a) $0 \leq \theta \leq \pi/2$ **(b)** $0 \leq \theta \leq 3\pi/4$

(c) $0 \leq \theta \leq 3\pi/2$ **(d)** $0 \leq \theta \leq 4\pi$

In Exercises 63–66, use the results of Exercise 61 to find the distance between the points with given polar coordinates.

63. (2, 10°), (5, 130°)

64. (4, 20°), (6, 65°)

65. (−3, 25°), (−5, 160°)

66. (6, −35°), (8, −65°)

Extending the Ideas

67. Graphing Polar Equations Parametrically Find parametric equations for the polar curve $r = f(\theta)$.

Group Activity In Exercises 68–71, use what you learned in Exercise 67 to write parametric equations for the given polar equation. Support your answers graphically.

68. $r = 2 \cos \theta$

69. $r = 5 \sin \theta$

70. $r = 2 \sec \theta$

71. $r = 4 \csc \theta$