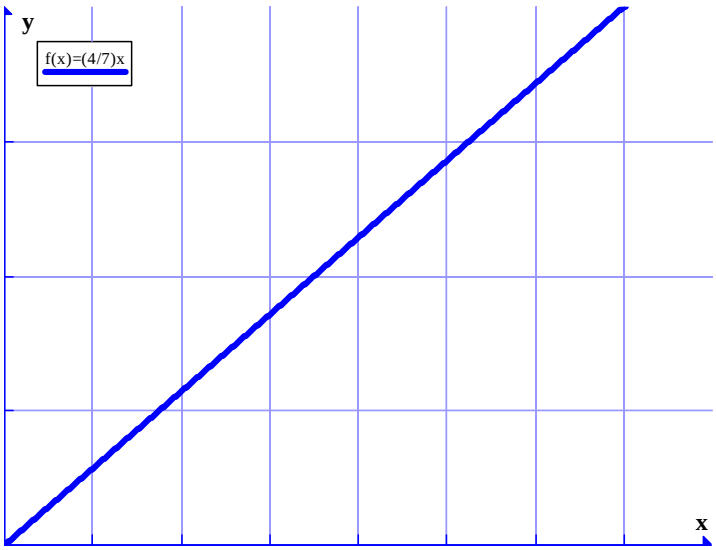
A New Look at Distances and Fractions

We are going to explore some mathematics used extensively by computers to approximate lines.

1. Consider the fraction $\frac{4}{7}$. This fraction can be representedwith the following graph:



a. Starting at (0, 0), mark a path on the graph from (0, 0) to (7, 4). For this path, you must follow these rules:
   - You can only move horizontally and vertically, by whole units.
   - You can never cross the segment.
   - You must "hug" the line as closely as possible.

b. Can you find another path that follows the rules? If so, draw it. If not, explain.

c. Let's use this notation to represent your paths: For each horizontal movement of 1 unit, write "0". For each vertical movement of 1 unit, write "1". (For example, the path 0010101 means you went over, over, up, over, up, over, up.) The "number" for your path is called the "Christoffel Word" for the slope of your line. You should have a "lower Christoffel word" and an "upper Christoffel word".

d. Find Christoffel words for each of the following fractions: $\frac{5}{8}$ and $\frac{5}{11}$. (Use the grid paper provided.)

e. Find the fraction that is represented by the Christoffel word: 01011011011

f. What patterns do you notice about the "upper" and "lower" paths and the "upper" and "lower" Christoffel words for each fraction?

g. Using your path for $\frac{4}{7}$, locate the point on the path that is closest to the line segment (exclude the beginning and ending points). We will "define" that distance as the vertical distance from the point to the segment. Verify that you have found the closest point by finding the vertical distance from each lattice point on the path to the segment. Call this point P.

h. Draw two line segments: one from (0, 0) to P, and one from P to the endpoint of your segment.

- Find the fraction representing each of these line segments.

  Describe the relationship between the fractions representing these two segments and the fraction representing the original line segment.

- Find the Christoffel words represented by each of these line segments.
  The resulting Christoffel words are called "factoring the sequence as the sum of two smaller sequences."

  Describe the relationship between the Christoffel words representing these two segments and the Christoffel word representing the original line segment.

2. Let's consider the path for the fraction $\frac{4}{7}$.

a. Use two different strategies to find the area between the "lower" path and the line segment from (0, 0) to (7, 4). Then use two strategies to find the area between the "upper" path and the line segment also.

b. Make a conjecture about which of the five fractions you "graphed" has the greatest area between the line segment and the path. Which has the least? Explain your reasoning.

c. Find a formula for finding the area between the line segment and the path for any reduced fraction $\frac{m}{n}$.

Write about this activity:

- What did you learn?

- How did it change your thinking?

- What new questions or "wonderings" would you still like to explore?