Calculus Chap. 7 section 5 Applications to science	Name: __________ Date: __________ Period: __________

1.

Find the work done by the force of $F(x)$ newtons along the x-axis from $x = a$ meters to $x = b$ meters.

$F(x) = x e^{-x/4}$, $a = 0$, $b = 7$

Work is defined as force times displacement. If the force $F(x)$ is not constant, then the work done in moving an object from $x = a$ to $x = b$ is the definite integral shown below.

$$W = \int_a^b F(x)\,dx$$

1. answer

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$$W = \int_a^b F(x)\,dx$$

Apply the definition of the work.

$$W = \int_a^b F(x)\,dx = \int_0^7 x e^{-x/4}\,dx$$

The integral can be evaluated using integration by parts. Let u = x and $dv = e^{-x/4}$. Differentiate u = x with respect to x.

$$du = dx$$

Use the rule $\int e^u du = e^u$ to integrate to find v.

$$v = \int e^{-x/4} dx = -4 e^{-x/4}$$

Apply the definition for integration by parts.

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ &= (x)\left(-4e^{-x/4}\right) - \int\left(-4e^{-x/4}\right)dx \end{aligned}$$

The remaining integral can now be evaluated.

$$(x)\left(-4e^{-x/4}\right) + 4\int e^{-x/4}dx = (x)\left(-4e^{-x/4}\right) + 4\left(-4e^{-x/4}\right)$$

Evaluate the results at the limits of integration.

1. answer cont.

$$(x)\left(-4e^{-x/4}\right)+4\left(-4e^{-x/4}\right) = \left[-4(x+4)\;e^{-x/4}\right]_0^7$$
$$\approx (-7.6461)-(-16)$$

Subtract to find the work done.

$(-7.6461)-(-16) \approx 8.354$

Therefore, approximately 8.354 joules of work are done by the force.

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2. It takes 1000 J of work to stretch a spring from its natural length of 1 m to a length of 2 m. Find the force constant of the spring.

Hooke's Law describes the relationship between the force, F, applied to a spring with spring constant k and the amount that the spring stretches, x. Symbolically, Hooke's Law is $F = kx$.

2. answer

It takes 1000 J of work to stretch a spring from its natural length of 1 m to a length of 2 m. Find the force constant of the spring.

Hooke's Law describes the relationship between the force, F, applied to a spring with spring constant k and the amount that the spring stretches, x. Symbolically, Hooke's Law is $F = kx$.

The work done by a variable force stretching a spring is $W = \int_{x_1}^{x_2} F(x)\,dx$, where x_1 represents the distance between the initial length and the natural length, and x_2 represents the distance between the final length and the natural length. In this exercise x_1 and x_2 are, respectively, 0 and 1.

Since $W = \int_{x_1}^{x_2} F(x)\,dx$, $F(x) = kx$, and $x_1 = 0$ m and $x_2 = 1$ m, $W = \int_0^1 kx\,dx = \left[\frac{1}{2}kx^2\right]_0^1$.

So, $W = 1000 = \left[\frac{1}{2}kx^2\right]_0^1 = \frac{1}{2}k(1)^2 = \frac{1}{2}k$.

Thus, the spring's force constant, k, is 2000 N / m.

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3.

It takes a force of 29,272 lb to compress a coil spring assembly from its free height of 6 inches to its fully compressed height of 2 inches.

a. What is the assembly's force constant?
b. How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in-lb.

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a. Hooke's law states that $F = kx$. Use this to determine the force constant k for the spring assembly.

3. answer

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a. Hooke's law states that $F = kx$. Use this to determine the force constant k for the spring assembly.

First, determine the distance the spring assembly is compressed by a force of 29,272 lb.

$6 - 2 = 4$ in

Substitute the known values in Hooke's law and solve for k.

$$\begin{aligned} F &= kx \\ 29{,}272 &= k(4) \\ 7318 &= k \end{aligned}$$

The force constant for the spring assembly is 7318 lb/in.

b. The work done by a variable force F(x) directed along the x-axis from $x = a$ to $x = b$ is given by the following definite integral.

$$W = \int_a^b F(x)\,dx$$

Picture the spring assembly laid out along the x-axis with its movable end at the origin and its fixed end at $x = 6$ inches. When the assembly is compressed the first half inch, the end moves from $x = 0$ inches to $x = 0.5$ inch.

To determine the work done to compress the assembly the first half inch, first substitute 0 for a, 0.5 for b, and 7318x for F(x). Then integrate.

$$\begin{aligned} \int_a^b F(x)\,dx &= \int_0^{0.5} 7318x\,dx \\ &= 3659x^2\Big]_0^{0.5} \end{aligned}$$

3. answer cont.

Now evaluate the definite integral.

$$3659x^2\Big]_0^{0.5} = 3659(0.5)^2 - 3659(0)^2$$
$$= 914.75$$
$$\approx 915 \text{ in-lb}$$

When the assembly is compressed the second half inch, the initial x-coordinate of the end of the spring assembly is $x = 0.5$ inch and the final x-coordinate is $x = 1$ inch.

Now evaluate the definite integral from $x = 0.5$ inch to $x = 1$ inch. Note that the integral for $x = 0.5$ in was evaluated previously.

$$3659x^2\Big]_{0.5}^{1} = 3659(1)^2 - 914.75$$
$$= 2744.25$$
$$\approx 2744 \text{ in-lb}$$

It takes about 915 in-lb of work to compress the spring assembly the first half inch and about 2744 in-lb of work to compress it the second half inch.

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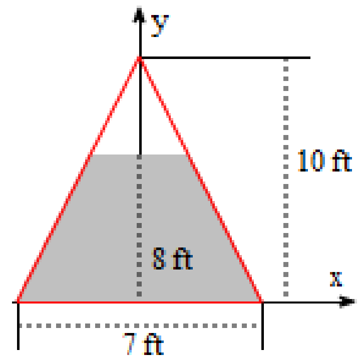
4. The vertical end of a tank containing molasses (grey shading) weighing $100.0\ \text{lb}/\text{ft}^3$ has the shape shown on the right. Find the force against the end of the tank as an integral and evaluate it.



The pressure, p, exerted by a fluid at a particular depth h is proportional to the depth of the fluid.

$p = wh$, where w is the weight density of the fluid.

4. answer

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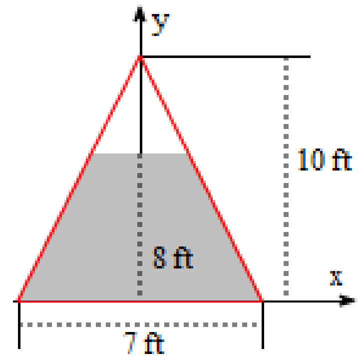

The pressure, p, exerted by a fluid at a particular depth h is proportional to the depth of the fluid.

$p = wh$, where w is the weight density of the fluid.

The force exerted by a fluid on a surface is related to the pressure as shown below.

$\text{Force} = \text{pressure} \times \text{area}$

Note that as the depth in the tank varies, both the pressure and the area on which the pressure is exerted at that depth vary as well. Therefore, the product of the pressure and the area must be integrated over the entire depth of the tank (along the y-axis) in order to find the force.

Write the pressure as a function of the variable of integration, y.

$p = 100.0(8 - y)$

At a particular depth, the fluid exerts a pressure equal to $100.0(8 - y)$ on a narrow horizontal strip of width Δy of the end of the tank. The area of this narrow strip is its length times its width.

The length of the strip at a particular value of y is given by two times the x-coordinate along one side of the tank. The equation of the line forming the right side of the tank is shown below.

$$y = -\frac{10}{3.5}x + 10$$

Rewrite this equation as a function, x, in terms of y.

4. answer cont.

$$x = \frac{3.5}{10}(10 - y)$$

The length of the segment at y is two times x.

$$\text{length} = \frac{7}{10}(10 - y)$$

So, the force exerted by the fluid on the narrow strip of the end of the tank is given in terms of y as shown below.

$$\text{Force} = 100.0(8 - y)\left(\frac{7}{10}(10 - y)\right)\Delta y$$

For $y = 0$ at the bottom of the tank, the surface of the molasses is at $y = 8$.

Integrate the force from the bottom of the tank to the surface of the molasses to find the total force on the tank wall.

$$\text{Force} = \int_0^8 100.0(8 - y)\left(\frac{7}{10}(10 - y)\right)dy$$

$$= \frac{100.0 \cdot 7}{10}\int_0^8 [8 \cdot 10 - (8 + 10)y + y^2]dy$$

$$= \frac{100.0 \cdot 7}{10}\left(80\,[y]_0^8 - 18\left[\frac{y^2}{2}\right]_0^8 + \left[\frac{y^3}{3}\right]_0^8\right)$$

Evaluate the integral.

Force = 16,426.7 lb (Rounded to the nearest tenth.)

Therefore, the molasses exerts a force of 16,426.7 lb on the tank wall.

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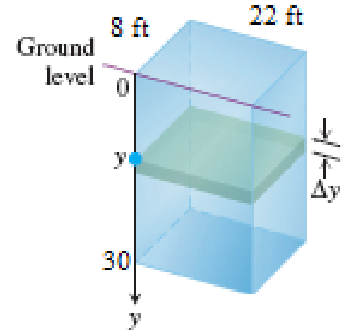
5.

The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.7 lb / ft^3.

a. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

b. If the water is pumped to ground level with a (6 / 11)-horsepower (hp) motor (work output 300 ft-lb/sec), how long will it take to empty the tank (to the nearest minute)?



a. Imagine the water divided into thin slabs by planes perpendicular to the y-axis. Each slab has a volume of $A \cdot \Delta y$, where A is the cross-sectional area of the tank and Δy is the thickness of the slab. The weight of each slab is $62.7 \cdot A \cdot \Delta y$. The work required to lift each slab y feet to ground level, where y is in the interval (0,30), is $62.7 \cdot A \cdot y \cdot \Delta y$. Use this information to create an integral.

5. answer

The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.7 lb / ft^3.
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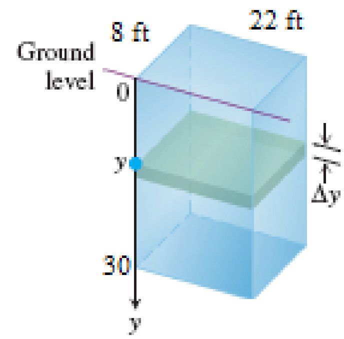

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The work of pumping all the water back to ground level is given by the definite integral below, where a is the initial height to ground level and b is the height to ground level from the bottom of the tank.

$$W = \int_a^b 62.7Ay\,dy$$

Calculate the cross-sectional area of the tank.

$$A = (8\text{ ft}) \cdot (22\text{ ft}) = 176\text{ ft}^2$$

The initial height to ground level when the tank is full is $a = 0$ feet.

The height to ground level from the bottom of the tank is $b = 30$ feet.

To determine the work done to pump all the water from the tank, first substitute 0 for a, 30 for b, and 176 for A. Then integrate.

$$\int_a^b 62.7Ay\,dy = \int_0^{30} 62.7(176)y\,dy$$

$$= 5517.6y^2\Big]_0^{30}$$

Now evaluate the integral.

5. answer cont.

$$5517.6y^2\Big]_0^{30} = 5517.6(30)^2 - 5517.6(0)^2$$
$$= 4{,}965{,}840 \text{ ft-lb}$$

It takes 4,965,840 ft-lb of work to empty the tank by pumping the water back to ground level once the tank is full.

b. Divide the work done by the rate of work to determine the time.

$$t = \frac{4{,}965{,}840 \text{ ft-lb}}{300 \text{ ft-lb / sec}}$$
$$= 16552.8 \text{ seconds}$$

Now convert from seconds to minutes.

$$(16552.8 \text{ sec}) \cdot \frac{1 \text{ min}}{60 \text{ sec}} = 275.88 \text{ minutes}$$

It takes about 276 minutes to empty the tank with a (6/11)-hp motor.

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6. A vertical right circular cylindrical tank measures 20 ft high and 8 ft in diameter. It is full of liquid weighing 50.6 lb / ft^3. How much work does it take to pump the liquid to the level of the top of the tank?

Imagine the liquid divided into thin slabs by planes perpendicular to the ground. Each slab has a volume of $A \cdot \Delta y$, where A is the cross-sectional area of the tank at y and Δy is the thickness of the slab. The weight of each slab is $50.6 \cdot A \cdot \Delta y$. The work required to lift each slab $(20 - y)$ feet to the top of the tank is $50.6 \cdot A \cdot (20 - y) \cdot \Delta y$. Use this information to create an integral.

6. answer

A vertical right circular cylindrical tank measures 20 ft high and 8 ft in diameter. It is full of liquid weighing 50.6 lb / ft^3. How much work does it take to pump the liquid to the level of the top of the tank?

Imagine the liquid divided into thin slabs by planes perpendicular to the ground. Each slab has a volume of $A \cdot \Delta y$, where A is the cross-sectional area of the tank at y and Δy is the thickness of the slab. The weight of each slab is $50.6 \cdot A \cdot \Delta y$. The work required to lift each slab $(20 - y)$ feet to the top of the tank is $50.6 \cdot A \cdot (20 - y) \cdot \Delta y$. Use this information to create an integral.

The work to pump all the liquid to the top of the tank is given by the definite integral below, where a is the height of the bottom of the tank and b is the height of the top of the tank.

$$W = \int_a^b 50.6A(20 - y)\, dy$$

The cross-sectional area of the tank is a circle. Write an expression for A.

$$A = 16\pi \text{ ft}^2$$

Find the height of the bottom of the tank

$$a = 0 \text{ ft}$$

Find the height of the top of the tank

$$a = 20 \text{ ft}$$

6. answer cont.

To determine the work done, first substitute 0 for a, 20 for b, and 16π for A. Then simplify.

$$\int_a^b 50.6A(20-y)\,dy = \int_0^{20} 50.6(16\pi)(20-y)\,dy$$
$$= 809.6\pi\int_0^{20}(20-y)\,dy$$

Now integrate.

$$809.6\pi\int_0^{20}(20-y)\,dy = 809.6\pi\left[20y-\frac{y^2}{2}\right]_0^{20}$$

Finally, evaluate the integral, rounding to the nearest whole number as needed.

$$809.6\pi\left[20y-\frac{y^2}{2}\right]_0^{20} = 809.6\pi\left[\left(20(20)-\frac{(20)^2}{2}\right)-\left(20(0)-\frac{(0)^2}{2}\right)\right]$$
$$\approx 508{,}687$$