Calculus
Chap. 7 section 1
Net Change part 2

Name: ______________________________
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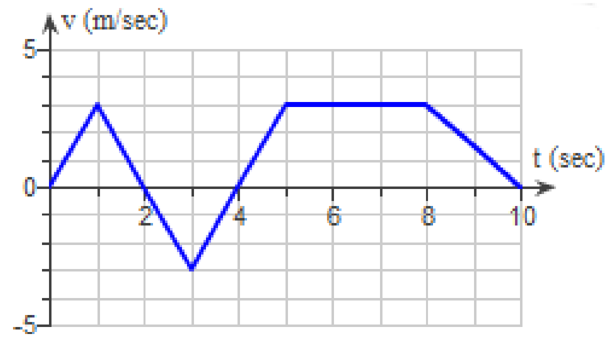
Period: ______________________________

1. The graph of the velocity of a particle moving on the x-axis is shown on the right. The particle starts at $x = 1$ when $t = 0$. Find where the particle is at the end of the trip and find the total distance traveled by the particle.



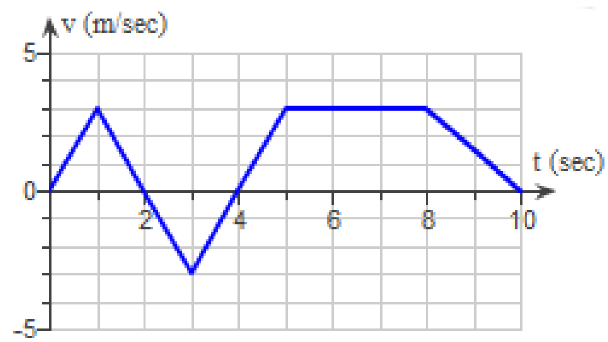
To find the position of the particle at the end of the trip, add its total displacement to its initial position.

Final position = initial position + displacement

The theory is started, you continue from here….

1. answer

The graph of the velocity of a particle moving on the x-axis is shown on the right. The particle starts at $x = 1$ when $t = 0$. Find where the particle is at the end of the trip and find the total distance traveled by the particle.

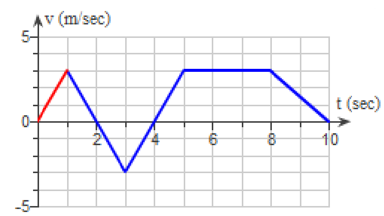


To find the position of the particle at the end of the trip, add its total displacement to its initial position.

Final position = initial position + displacement

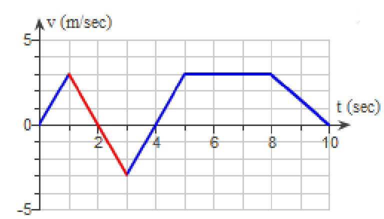
The theory is started, you continue from here….

The displacement of the particle during a particular time interval $[t_1, t_2]$ is given by the integral of the velocity function over that time interval.



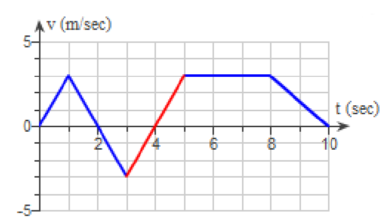
$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\,dt$$

Note that velocity function has several forms throughout the trip. From $t = 0$ to $t = 1$, the velocity starts at zero and increases at a constant rate of $3 \text{ m} / \text{sec}^2$.



Then, from $t = 1$ to $t = 3$, the velocity decreases at a constant rate of $-3 \text{ m} / \text{sec}^2$.

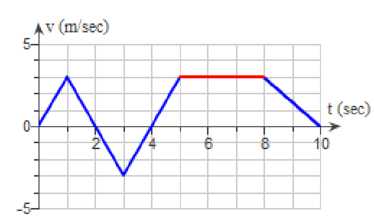
From $t = 3$ to $t = 5$, the velocity increases at a constant rate of $3 \text{ m} / \text{sec}^2$.

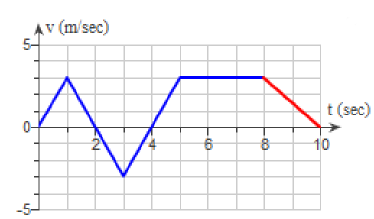


From $t = 5$ to $t = 8$, the velocity remains constant at $3 \text{ m} / \text{sec}$.

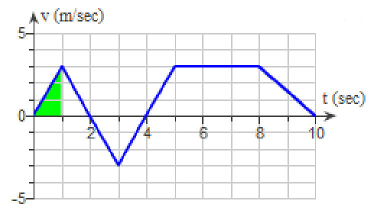
Finally, from $t = 8$ to $t = 10$, the velocity decreases at a constant rate of $-1.5 \text{ m} / \text{sec}^2$.

Find the displacement during each of these time intervals by calculating $\int_{t_1}^{t_2} v(t)\,dt$ for each interval.



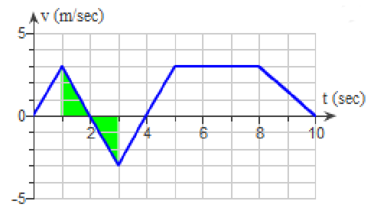


1. answer cont.



For the velocity function shown, recall that the integral of a function corresponds to the area enclosed by the graph of the function and the x-axis, as shown below.

$$\int_a^b f(x)\,dx = (\text{area above the x-axis}) - (\text{area below the x-axis})$$

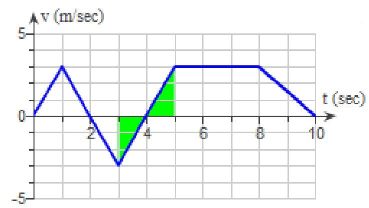


The region enclosed the velocity graph and the x-axis in the interval between $t = 1$ and $t = 3$ is a triangle of base width 1 and height 3.

The displacement during this interval is equal to the area of this triangle.

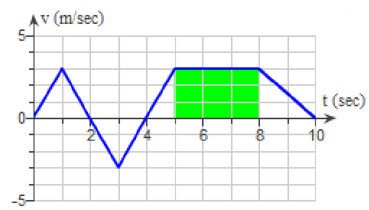
$$\text{Displacement} = \frac{1}{2}bh$$

$$= \frac{1}{2} \cdot 1 \cdot 3 = 1.5 \text{ m}$$



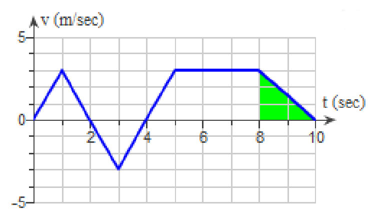
To calculate the displacement from $t = 1$ to $t = 3$, note that the area of the triangle above the x-axis is equal to the area of the triangle below the x-axis.

This corresponds to a displacement in one direction followed by an equal displacement in the opposite direction, resulting in a total displacement of zero for this time interval.



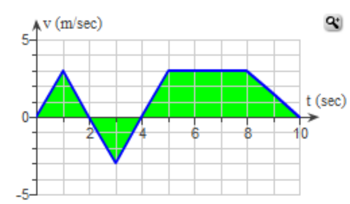
$$\text{Displacement} = \int_1^3 v(t)\,dt = 0 \text{ m}$$

The displacement from $t = 3$ to $t = 5$ is also zero because the areas above and below the x-axis are equal.

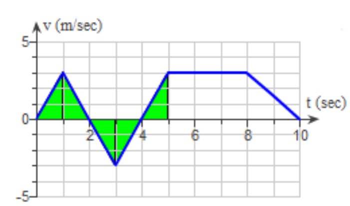


$$\text{Displacement} = \int_3^5 v(t)\,dt = 0 \text{ m}$$

The displacement from $t = 5$ to $t = 8$ is the area of the rectangle between the velocity graph and the x-axis. This rectangle has a base width of 3 and a height of 3.



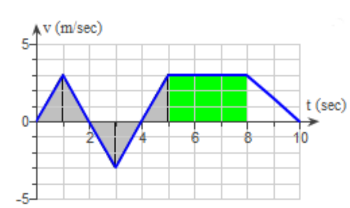
$$\text{Displacement} = \int_5^8 v(t)\,dt = 9 \text{ m}$$



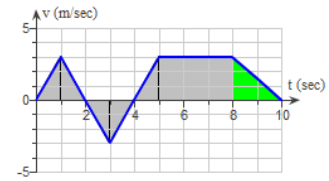
The displacement during the interval from $t = 8$ to $t = 10$ is the area of the triangle between the velocity graph and the x-axis in this interval.

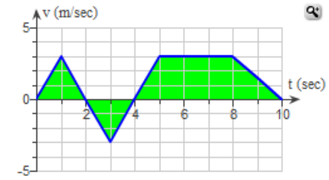
$$\text{Displacement} = \frac{1}{2}bh$$

$$= \frac{1}{2} \cdot 2 \cdot 3 = 3 \text{ m}$$



1. answer cont.





Add the total displacement over all of the time intervals to the initial position of the particle to find its final position.

Final position = 1 + (1.5 + 0 + 0 + 9 + 3) = 14.5 m

The total distance traveled by the particle during a particular time interval $[t_1, t_2]$ is given by the integral of the absolute value of the velocity function over that interval.

$$\text{Total distance traveled} = \int_{t_1}^{t_2} |v(t)|\,dt$$

In terms of the area between the graph and the x-axis, this means that all the areas calculated for the displacement, whether they are above or below the x-axis, are treated as positive distances when calculating the total distance traveled.

The total area enclosed between the velocity graph and the x-axis includes five triangles of base width 1 and height 3.

These triangles have the combined area shown below.

$$\text{Area} = 5 \cdot \frac{1}{2}bh$$

$$= 5 \cdot \frac{1}{2} \cdot 1 \cdot 3 = 7.5$$

Between t = 5 and t = 8, the rectangle of base width 3 and height 3 has an area of 9, as previously calculated.

Finally, add the area of the triangle in the interval between t = 8 and t = 10. This triangle's area is 3, as previously calculated.

The sum of the areas is shown below.

$7.5 + 9 + 3 = 19.5$

Therefore, the total distance traveled by the particle is 19.5 meters.

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2. Data from an independent research company found that the annual cost per worker for insurance (health, life, liability, etc.) was increasing according to the function $f(x) = 71.72\,e^{0.34x}$, where f(x) is the cost in dollars at time x, and x is the number of years measured from the beginning of the year 2000. That is, x = 0 corresponds to the start of 2000. Find the total increase in costs during the next 5 years, beginning in 2000.

To find the total increase in costs, evaluate the definite integral of the function over the number of years the function was increasing.

2. answer

Data from an independent research company found that the annual cost per worker for insurance (health, life, liability, etc.) was increasing according to the function $f(x) = 71.72\,e^{0.34x}$, where f(x) is the cost in dollars at time x, and x is the number of years measured from the beginning of the year 2000. That is, x = 0 corresponds to the start of 2000. Find the total increase in costs during the next 5 years, beginning in 2000.

To find the total increase in costs, evaluate the definite integral of the function over the number of years the function was increasing.

To calculate the definite integral of f over [a,b], find an antiderivative F of f, and calculate the number $\int_a^b f(x)\,dx = F(b) - F(a)$. The usual notation for $F(b) - F(a)$ is $\left[F(x)\right]_a^b$.

Begin by finding F, the antiderivative of the function $71.72\,e^{0.34x}$.

$$\int_0^5 71.72\,e^{0.34x}\,dx = \left(\frac{71.72}{0.34}\,e^{0.34x}\right)\Bigg|_0^5$$

Now evaluate the definite integral using $F = \frac{71.72}{0.34}\,e^{0.34x}$. Substitute in the upper and lower limits of the integral.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_0^5 71.72\,e^{0.34x}\,dx = \left[\frac{71.72}{0.34}\,e^{0.34(5)}\right] - \left[\frac{71.72}{0.34}\,e^{0.34(0)}\right]$$

Evaluate $\frac{71.72}{0.34}\,e^{0.34(5)} - \frac{71.72}{0.34}\,e^{0.34(0)}$ by simplifying each term.

$$\begin{aligned}\int_0^5 71.72\,e^{0.34x}\,dx &= \frac{71.72}{0.34}\,e^{0.34(5)} - \frac{71.72}{0.34}\,e^{0.34(0)} \\ &= 943.74 \qquad \text{(rounded to the nearest hundredth)}\end{aligned}$$

Thus, the total increase in costs is \$943.74.

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3. Population density measures the number of people per square mile inhabiting a given living area. A certain city's population density, which decreases as you move away from the city center, can be approximated by the function $15{,}000(4-r)$ at a distance r miles from the city center. Use this information to answer the following questions.

(a) If the population density approaches zero at the edge of the city, what is the city's radius?

3. answer

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(a) If the population density approaches zero at the edge of the city, what is the city's radius?

Set the expression for the population density equal to 0 and solve for r.

$$15{,}000(4 - r) = 0$$
$$r = 4$$

The city's radius is 4 miles.

(b) A thin ring around the center of the city has thickness Δr and radius r. If you straighten it out, it suggests a rectangular strip. Approximately what is its area?

One dimension of the rectangle is the thickness Δr. The other dimension is the circumference of the circle.

The area of the rectangle is the product of the circumference $2\pi r$ and Δr. The area is the length times the width.

Therefore, the area of a thin ring is approximately $2\pi r \Delta r$.

(c) Why is the population of the thin ring approximately $15{,}000(4 - r)(2\pi r)\Delta r$?

The population density is approximately constant on the ring, so the population is the product of the density and the area.

(d) Estimate the total population of the city by setting up and evaluating a definite integral.

The population of the kth thin ring is $15{,}000(4 - r_k)(2\pi r_k)\Delta r$. As Δr goes to zero, the sum of the population of each ring approaches the integral shown below.

$$\text{Population} = \int_{r_1}^{r_2} 15{,}000(4 - r)(2\pi r)\, dr$$

3. answer cont.

The lower limit is the radius at the center of the city, or 0 miles. The upper limit is the radius at the edge of the city, or 4 miles.

Simplify the integrand and use antiderivatives to integrate.

$$30{,}000\pi \int_0^4 (4r - r^2)\,dr = 30{,}000\pi \left[2r^2 - \frac{1}{3}r^3\right]_0^4$$

Evaluate the result at each point.

$$30{,}000\pi \left[2r^2 - \frac{1}{3}r^3\right]_0^4 = 30{,}000\pi \left[\frac{32}{3} - 0\right]$$

Simplify to find the population.

$$30{,}000\pi \left[\frac{32}{3} - 0\right] \approx 1{,}005{,}310 \text{ people}$$

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4. A machine fills milk cartons with milk at an approximately constant rate, but backups along the assembly line cause some variation. The rates (in cases per hour) are recorded at hourly intervals during a 7-hour period, from 8:00 am to 3:00 pm, and are shown below. Use the trapezoidal rule with $n = 7$ to determine approximately how many cases of milk were filled by the machine over the 7-hour period.

Time	8	9	10	11	12	1	2	3
Rate (cases/h)	123	121	112	110	117	117	116	120

First approximate how many cases of milk were filled by the machine over the 7-hour period as a Riemann sum of values of a continuous function multiplied by interval lengths. If $f(x)$ is the function and $[a, b]$ the interval, and the interval is partitioned into subintervals of length Δx, the approximating sums will have the form $\sum f(c_k)\Delta x$ with c_k a point in the kth subinterval.

4. answer

A machine fills milk cartons with milk at an approximately constant rate, but backups along the assembly line cause some variation. The rates (in cases per hour) are recorded at hourly intervals during a 7-hour period, from 8:00 am to 3:00 pm, and are shown below. Use the trapezoidal rule with $n = 7$ to determine approximately how many cases of milk were filled by the machine over the 7-hour period.

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Let R(t), $0 \le t \le 7$, be the filling rate as a continuous function of time for the 7-hour period. Partition the 7-hour period into short subintervals of length Δt on which the rate is nearly constant and form the sum $\sum R(t_k)\Delta t$ as an approximation to the amount filled during the 7-hour period.

Next write a definite integral to express the limit of these sums as the norms of the partitions go to zero. Determine the integral formula for the number of cartons filled.

$$\int_0^7 R(t)\,dt$$

The trapezoidal rule states that to approximate $\int_a^b f(x)\,dx$, use

$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$, where [a, b] is partitioned into n subintervals of equal length $h = \frac{b-a}{n}$.

Since we have no formula for R in this instance and the values are equally spaced, use the trapezoidal rule to estimate the integral $\int_0^7 R(t)\,dt$. Recall that the length of the subintervals is 1 hour and that the rates are recorded during a 7-hour period.

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

$$\int_0^7 R(t)\,dt \approx \frac{7}{2 \cdot 7}[123 + 2(121) + 2(112) + 2(110) + 2(117) + 2(117) + 2(116) + 120]$$

$$\approx 815$$

Therefore, approximately 815 cases of milk were filled by the machine over the 7-hour period.

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5. A certain spring requires a force of 18 N to stretch it 3 cm beyond its natural length.

(a) What force would be required to stretch the spring 10 cm beyond its natural length?

(b) What would be the work done in stretching the spring 10 cm beyond its natural length?

(a) Hooke's Law for springs says that the force it takes to stretch or compress a spring x units from its natural length is a constant times x, $F = kx$.

5. answer

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(b) What would be the work done in stretching the spring 10 cm beyond its natural length?

(a) Hooke's Law for springs says that the force it takes to stretch or compress a spring x units from its natural length is a constant times x, $F = kx$.

Use the given information to find the constant k. Substitute the known values into the equation for the force and solve for k.

$$F = kx$$
$$18 = k(3)$$

Solve for the spring constant k.

$$18 = 3k$$
$$k = 6 \text{ N/cm}$$

Now use the equation to find the force required to stretch the spring 10 cm.

$$\begin{aligned} F &= kx \\ &= (6 \text{ N/cm})(10 \text{ cm}) \\ &= 60 \text{ N} \end{aligned}$$

(b) Let Δx be a short displacement and x_k be the position of the kth interval. Assumin the force is constant over the interval, the work done is the force times the displaceme $F(x_k)\Delta x$.

The sum $\sum F(x_k)\Delta x$ approximates the net work done by F over an interval. As Δx g zero, the sum converges to the integral of the force over the interval.

$$\text{Work} = \int_{x_1}^{x_2} F(x)\, dx$$

Use an antiderivative to evaluate the integral.

$$\int_0^{10} 6x\, dx = \left[3x^2\right]_0^{10}$$

Evaluate the result at each point.

$$\left[3x^2\right]_0^{10} = (300) - (0)$$

Subtract to find the work.

$$(300) - (0) = 300 \text{ N} \cdot \text{cm}$$

Finally, convert the work to Joules.

$$300 \text{ N} \cdot \text{cm} = 3 \text{ J}$$