Calculus	Name: ________________________
Chap. 6 section 5	Date: ________________________
Exponential growth & decay	Period: ________________________

Use separation of variables to solve the initial value problem. Indicate the domain over which the solution is valid.

$\frac{dy}{dx} = 3e^{x-y}$ and $y = 5$ when $x = 0$

A differential equation of the form $\frac{dy}{dx} = f(y)g(x)$ is called separable. Separate the variables by writing the equation in the form $\frac{1}{f(y)}dy = g(x)dx$. The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

The theory is started, you continue from here….

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Since $3e^{x-y}$ can be rewritten as $3e^{x}e^{-y}$, the equation is separable because it can be written in the form $\frac{dy}{dx} = f(y)g(x)$, where $f(y) = e^{-y}$ and $g(x) = 3e^{x}$.

Separate the variables.

$$\frac{dy}{dx} = 3e^{x}e^{-y}$$
$$\frac{1}{e^{-y}}dy = 3e^{x}dx$$
$$e^{y}dy = 3e^{x}dx$$

Antidifferentiate both sides of the equation.

$$\int e^{y}dy = \int 3e^{x}dx$$
$$e^{y} = 3e^{x} + C$$

Note that only one constant is needed.

Then, apply the initial condition to find C. Note that the initial condition is $y = 5$ when $x = 0$.

$$e^{5} = 3e^{0} + C$$
$$e^{5} = 3 + C$$
$$e^{5} - 3 = C$$

Substitute for C, and solve for y.

$$e^{y} = 3e^{x} + e^{5} - 3$$
$$\ln(e^{y}) = \ln(3e^{x} + e^{5} - 3)$$
$$y = \ln(3e^{x} + e^{5} - 3)$$

This solution is valid for the continuous section of the function that goes through the point (0,5), that is, on the domain $(-\infty,\infty)$.

Complete the table for an investment if interest is compounded continuously.

Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
2500	6.2		

The formula for compounding interest continuously is $A(t) = A_0 e^{rt}$, where A(t) is the amount after time t, A_0 is the initial amount, r is the rate, and t is the amount of time.

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To find the doubling time, set A(t) to be twice A_0 in the formula for compounding interest continuously, $A(t) = A_0 e^{rt}$, and solve for t. Express the interest rate as a decimal.

$$A(t) = A_0 e^{rt}$$
$$5000 = 2500 e^{(0.062)t}$$

Divide both sides of the equation by 2500 and then take the natural log of both sides.

$$2 = e^{(0.062)t}$$
$$\ln 2 = \ln\left(e^{(0.062)t}\right)$$
$$\ln 2 = (0.062)t$$

Now divide both sides of the equation by 0.062 to solve for t.

Doubling time ≈ 11.2 years

Next use the formula for compounding interest continuously, $A(t) = A_0 e^{rt}$, to find the amount after 30 years.

$$A(30) = 2500 e^{0.062(30)}$$
$$\approx \$16{,}059.34$$

The completed table is shown below.

Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
2500	6.2	11.2	16,059.34

Complete the table for an investment if interest is compounded continuously.

Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
	3.75		308.02

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Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
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The formula for compounding interest continuously is shown below, where A(t) is the amount after t years if an initial amount A_0 is invested at a fixed annual interest rate r (expressed as a decimal).

$$A(t) = A_0 e^{rt}$$

To find the initial deposit, use the fact that the rate, expressed as a decimal, is 0.0375 and A(30) is $308.02. Substitute the values in the formula and solve for A_0, rounded to the nearest dollar.

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 308.02 &= A_0 e^{(0.0375)(30)} \\ 308.02 &= A_0 e^{1.125} \\ 100 &= A_0 \end{aligned}$$

To find the doubling time, use the fact that, when the value has doubled, A(t) is twice A_0.

Substitute $2A_0$ for A(t) and 0.0375 for r in the formula and solve for t, rounded to the nearest tenth of a year.

$$\begin{aligned} A(t) &= A_0 e^{rt} \\ 2A_0 &= A_0 e^{0.0375t} \\ 2 &= e^{0.0375t} \\ 18.5 &= t \end{aligned}$$

The initial amount is $100 and it takes about 18.5 years for the investment to double in value. The completed table is shown below.

Initial Deposit ($)	Annual Rate (%)	Doubling Time (yr)	Amount in 30 yr ($)
100	3.75	18.5	308.02

Suppose that P is invested in a savings account in which interest, k, is compounded continuously at 3% per year. The balance P(t) after time t, in years, is $P(t) = Pe^{kt}$.

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a) The exponential growth function will be in the form $P(t) = Pe^{kt}$.

You are to leave P in the function, since initially no investment amount is given.

In this case, $k = 0.03$.

The growth function is thus $P(t) = Pe^{0.03t}$.

b) If \$2000 in invested, P = \$2000.

The growth function is now $P(t) = 2000e^{0.03t}$.

To find the balance after 2 years, evaluate P(t) for $t = 2$.

$$P(2) = 2000e^{0.03 \cdot 2}$$

$$= 2000e^{0.06}$$

$$\approx 2000(1.06183655)$$

$$\approx 2123.67$$

The balance after 2 years is \$ 2123.67.

c) To find the number of years after which an investment of \$2000 will double itself, set $P(t) = 4000$ and solve for t.

$$4000 = 2000e^{0.03t}$$

$$\frac{4000}{2000} = e^{0.03t}$$

$$2 = e^{0.03t}$$

How should you solve for t?

Take the logarithm of both sides.

Take the natural logarithm on both sides, since the base of the exponential function is *e*.

$$\ln 2 = \ln (e^{0.03t})$$

Since $\log_a a^k = k$, $\ln 2 = 0.03t$ and $t = \frac{\ln 2}{0.03}$.

$$t \approx \frac{0.6931}{0.03}$$

$$t \approx 23.1$$

An investment of \$2000 will double itself in about 23.1 years.

The radioactive decay of an isotope can be modeled by the differential equation $dy/dt = -0.0091y$, where t is measured in years. Find the half-life of the isotope.

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The decay of a radioactive substance is described by the equation $dy/dt = -ky$, with the rate constant $k > 0$.

In the differential equation that models the radioactive decay of the isotope, the value that corresponds to k is 0.0091.

The half-life of the substance is $\frac{\ln 2}{k}$.

Calculate the half-life, rounded to the nearest year.

$$\begin{aligned} \frac{\ln 2}{k} &= \frac{\ln 2}{0.0091} \\ &= 76 \end{aligned}$$

The half-life of the isotope is about 76 years.

The decay equation for a particular radioactive isotope is $y = y_0 e^{-0.1t}$, t in days. Determine the time required for a confined sample of the isotope to fall to 40% of its original value.

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To determine the time required for a confined sample of the isotope to fall to 40% of its original value, solve the decay equation for t when $y = 0.4y_0$.

To solve $0.4y_0 = y_0 e^{-0.1t}$, divide both sides by y_0 and then take the natural logarithm of both sides.

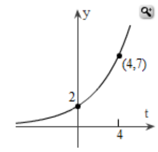
$$\frac{0.4y_0}{y_0} = \frac{y_0}{y_0} e^{-0.1t}$$

$$0.4 = e^{-0.1t}$$

$$\ln 0.4 = -0.1t$$

Solving $\ln 0.4 = -0.1t$ for t gives the time as 9.16 days.

Find the exponential function $y = y_0 e^{kt}$ whose graph passes through the two points.



Use the two points given in the graph and the general form of an exponential function, $y = y_0 e^{kt}$, to solve for y_0 and k. Then use these values to write the explicit exponential function.

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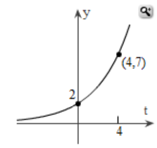

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The first point is (0,2). Use this point to solve for y_0. Substitute 0 for t and 2 for y in the function $y = y_0 e^{kt}$ and solve for y_0.

$$y = y_0 e^{kt}$$
$$(2) = y_0 e^{k(0)}$$
$$2 = y_0 \qquad a^0 = 1$$

The function can now be written in the form $y = 2e^{kt}$. Use the second point, (4,7), to solve for k.

$$y = 2e^{kt}$$
$$(7) = 2e^{k(4)}$$
$$7/2 = e^{4k} \qquad \text{Divide both sides of the equation by 2.}$$

Next take the natural log of both sides and solve for k.

$$\ln(7/2) = \ln(e^{4k})$$
$$\ln(7/2) = 4k \qquad \ln(e^a) = a$$
$$0.313 \approx k$$

Thus, the exponential function that passes through the two points is $y = 2e^{0.313t}$.

According to Newton's Law of Cooling, if a body with temperature T_1 is placed in surroundings with temperature T_0, different from that of T_1, the body will either cool or warm to temperature T(t) after t minutes, where $T(t) = T_0 + (T_1 - T_0)e^{-kt}$.

A cup of coffee with temperature 150°F is placed in a freezer with temperature 0°F. After 10 minutes, the temperature of the coffee is 67°F. Use Newton's Law of Cooling to find the coffee's temperature after 15 minutes.

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The temperature of the freezer is 0°F, so $T_0 = 0$.

$T_1 = 150$

Substitute these values into the function and simplify.

$T(t) = 0 + (150 - 0)e^{-kt}$

$T(t) = 150e^{-kt}$

Since the temperature cools to 67°F in 10 minutes, you know $T(10) = 67$. Use this to determine the constant, k.

$T(10) = 150e^{-k(10)}$

$67 = 150e^{-10k}$

Divide by 150.

$$\frac{67}{150} = e^{-10k}$$

Take the natural log of both sides.

$$\ln\frac{67}{150} = \ln e^{-10k}$$

Simplify the right side of the equation.

$$\ln\frac{67}{150} = -10k$$

To solve for k, divide both sides by -10 and evaluate the expression.

$$k = \frac{\ln\frac{67}{150}}{-10} \approx 0.08059$$

Substituting the value for k, the function becomes $T(t) = 150\,e^{-0.08059t}$.

Now find the temperature of the coffee after 15 minutes. Let $t = 15$ in the function.

$T(15) = 150\,e^{-0.08059(15)}$

$T(15) = 45$

Therefore, after 15 minutes the coffee will have a temperature of 45°F.