Calculus
Chap. 6 section 2
U substitution

Name: ______________________________

Date: ______________________________

Period: ______________________________

So

$$\int_0^1 \frac{x}{x^2-4}\,dx = \frac{1}{2}\int_0^1 \frac{2x\,dx}{x^2-4}$$

$$= \frac{1}{2}\int_{-4}^{-3} \frac{du}{u} \qquad \text{Substitute } u\text{-interval for } x\text{-interval.}$$

$$= \frac{1}{2}\ln|u|\Big|_{-4}^{-3}$$

$$= \frac{1}{2}(\ln 3 - \ln 4) = \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

Notice that $\ln u$ would not have existed over the interval of integration $[-4, -3]$. The absolute value in the antiderivative is important. *Now try Exercise 63.*

Finally, consider this historical note. The technique of u-substitution derived its importance from the fact that it was a powerful tool for antidifferentiation. Antidifferentiation derived its importance from the Fundamental Theorem, which established it as the way to evaluate definite integrals. Definite integrals derived their importance from real-world applications. While the applications are no less important today, the fact that the definite integrals can be easily evaluated by technology has made the world less reliant on antidifferentiation, and hence less reliant on u-substitution. Consequently, you have seen in this book only a sampling of the substitution tricks calculus students would have routinely studied in the past. You may see more of them in a differential equations course.

Quick Review 6.2 *(For help, go to Sections 3.6 and 3.9.)*

In Exercises 1 and 2, evaluate the definite integral.

1. $\int_0^2 x^4\,dx$
2. $\int_1^5 \sqrt{x-1}\,dx$

In Exercises 3–10, find dy/dx.

3. $y = \int_2^x 3^t\,dt$
4. $y = \int_0^x 3^t\,dt$
5. $y = (x^3 - 2x^2 + 3)^4$
6. $y = \sin^2(4x - 5)$
7. $y = \ln \cos x$
8. $y = \ln \sin x$
9. $y = \ln(\sec x + \tan x)$
10. $y = \ln(\csc x + \cot x)$

Section 6.2 Exercises

In Exercises 1–6, find the indefinite integral.

1. $\int (\cos x - 3x^2)\,dx$
2. $\int x^{-2}\,dx$
3. $\int \left(t^2 - \frac{1}{t^2}\right)dt$
4. $\int \frac{dt}{t^2+1}$
5. $\int (3x^4 - 2x^{-3} + \sec^2 x)\,dx$
6. $\int (2e^x + \sec x \tan x - \sqrt{x})\,dx$

In Exercises 7–12, use differentiation to verify the antiderivative formula.

7. $\int \csc^2 u\,du = -\cot u + C$
8. $\int \csc u \cot u = -\csc u + C$
9. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$
10. $\int 5^x\,dx = \frac{1}{\ln 5}5^x + C$
11. $\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$
12. $\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$

In Exercises 13–16, verify that $\int f(u)\,du \neq \int f(u)\,dx$

13. $f(u) = \sqrt{u}$ and $u = x^2$ $(x > 0)$

14. $f(u) = u^2$ and $u = x^3$

15. $f(u) = e^u$ and $u = 7x$

16. $f(u) = \sin u$ and $u = 4x$

In Exercises 17–24, use the indicated substitution to evaluate the integral. Confirm your answer by differentiation.

17. $\int \sin 3x\,dx, \quad u = 3x$

18. $\int x \cos(2x^2)\,dx, \quad u = 2x^2$

19. $\int \sec 2x \tan 2x\,dx, \quad u = 2x$

20. $\int 28(7x - 2)^3\,dx, \quad u = 7x - 2$

21. $\int \frac{dx}{x^2 + 9}, \quad u = \frac{x}{3}$

22. $\int \frac{9r^2\,dr}{\sqrt{1 - r^3}}, \quad u = 1 - r^3$

23. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

24. $\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy, \quad u = y^4 + 4y^2 + 1$

In Exercises 25–46, use substitution to evaluate the integral.

25. $\int \frac{dx}{(1 - x)^2}$

26. $\int \sec^2(x + 2)\,dx$

27. $\int \sqrt{\tan x}\,\sec^2 x\,dx$

28. $\int \sec\left(\theta + \frac{\pi}{2}\right)\tan\left(\theta + \frac{\pi}{2}\right)d\theta$

29. $\int \tan(4x + 2)\,dx$

30. $\int 3(\sin x)^{-2}\,dx$

31. $\int \cos(3z + 4)\,dz$

32. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

33. $\int \frac{\ln^6 x}{x}\,dx$

34. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

35. $\int s^{1/3}\cos(s^{4/3} - 8)\,ds$

36. $\int \frac{dx}{\sin^2 3x}$

37. $\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)}\,dt$

38. $\int \frac{6\cos t}{(2 + \sin t)^2}\,dt$

39. $\int \frac{dx}{x \ln x}$

40. $\int \tan^2 x \sec^2 x\,dx$

41. $\int \frac{x\,dx}{x^2 + 1}$

42. $\int \frac{40\,dx}{x^2 + 25}$

43. $\int \frac{dx}{\cot 3x}$

44. $\int \frac{dx}{\sqrt{5x + 8}}$

45. $\int \sec x\,dx$ (*Hint:* Multiply the integrand by

$$\frac{\sec x + \tan x}{\sec x + \tan x}$$

and then use a substitution to integrate the result.)

46. $\int \csc x\,dx$ (*Hint:* Multiply the integrand by

$$\frac{\csc x + \cot x}{\csc x + \cot x}$$

and then use a substitution to integrate the result.)

In Exercises 47–52, use the given trigonometric identity to set up a u-substitution and then evaluate the indefinite integral.

47. $\int \sin^3 2x\,dx, \quad \sin^2 2x = 1 - \cos^2 2x$

48. $\int \sec^4 x\,dx, \quad \sec^2 x = 1 + \tan^2 x$

49. $\int 2\sin^2 x\,dx, \quad \cos 2x = 2\sin^2 x - 1$

50. $\int 4\cos^2 x\,dx, \quad \cos 2x = 1 - 2\cos^2 x$

51. $\int \tan^4 x\,dx, \quad \tan^2 x = \sec^2 x - 1$

52. $\int (\cos^4 x - \sin^4 x)\,dx, \quad \cos 2x = \cos^2 x - \sin^2 x$

In Exercises 53–66, make a u-substitution and integrate from $u(a)$ to $u(b)$.

53. $\int_0^3 \sqrt{y + 1}\,dy$

54. $\int_0^1 r\sqrt{1 - r^2}\,dr$

55. $\int_{-\pi/4}^{0} \tan x \sec^2 x\,dx$

56. $\int_{-1}^{1} \frac{5r}{(4 + r^2)^2}\,dr$

57. $\int_0^1 \frac{10\sqrt{\theta}}{(1 + \theta^{3/2})^2}\,d\theta$

58. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4 + 3\sin x}}\,dx$

59. $\int_0^1 \sqrt{t^5 + 2t}\,(5t^4 + 2)\,dt$

60. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

61. $\int_0^3 \frac{dx}{x + 2}$

62. $\int_2^5 \frac{dx}{2x - 3}$

63. $\int_1^2 \frac{dt}{t - 3}$

64. $\int_{\pi/6}^{2\pi/3} \cot x\,dx$

65. $\int_{-1}^{3} \frac{x\,dx}{x^2 + 1}$

66. $\int_0^2 \frac{e^x\,dx}{3 + e^x}$

8. $u = y^4 + 4y^2 + 1$

$du = (4y^3 + 8y)\,dy$

$du = 4(y^3 + 2y)\,dy$

$\frac{1}{4}\,du = (y^3 + 2y)\,dy$

$$\int 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)\,dy = 8\left(\frac{1}{4}\right)\int u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(y^4 + 4y^2 + 1)^3 + C$$

Check: $\frac{d}{dx}\left[\frac{2}{3}(y^4 + 4y^2 + 1)^3 + C\right]$

$= 2(y^4 + 4y^2 + 1)^2(4y^3 + 8y)$

$= 8(y^4 + 4y^2 + 1)^2(y^3 + 2y)$

9. Let $u = 1 - x$

$du = -dx$

$$\int \frac{dx}{(1-x)^2} = -\int \frac{du}{u^2} = u^{-1} + C = \frac{1}{1-x} + C$$

10. Let $u = x + 2$

$du = dx$

$$\int \sec^2(x+2)\,dx = \int \sec^2 u\,du = \tan u + C = \tan(x+2) + C$$

11. Let $u = \tan x$

$du = \sec^2 x\,dx$

$$\int \sqrt{\tan x}\sec^2 x\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

12. Let $u = \theta + \frac{\pi}{2}$

$du = d\theta$

$$\int \sec\left(\theta + \frac{\pi}{2}\right)\tan\left(\theta + \frac{\pi}{2}\right)d\theta = \int \sec u \tan u\,du = \sec u + C = \sec\left(\theta + \frac{\pi}{2}\right) + C$$

13. Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{du}{u} = \ln|u|\Big]_1^{\ln 6} = \ln(\ln 6)$$

14. Let $u = \tan x$

$du = \sec^2 x\,dx$

$$\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^2 x\,dx = \int_{-1}^{1} u^2\,du = \frac{1}{3}u^3\Big]_{-1}^{1} = \frac{1}{3}(1) - \frac{1}{3}(-1)^3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

15. Let $u = 3z + 4$

$du = 3\,dz$

$\frac{1}{3}\,du = dz$

$$\int \cos(3z+4)\,dz = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3z+4) + C$$

16. Let $u = \cot x$

$du = -\csc^2 x\,dx$

$$\int \sqrt{\cot x}\csc^2 x\,dx = -\int u^{1/2}\,du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C$$

17. Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{\ln^6 x}{x}\,dx = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}(\ln^7 x) + C$$

18. Let $u = \tan\frac{x}{2}$

$du = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$$\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx = 2\int u^7\,du = 2\cdot\frac{1}{8}u^8 + C = \frac{1}{4}\tan^8\frac{x}{2} + C$$

19. Let $u = s^{4/3} - 8$

$du = \frac{4}{3}s^{1/3}\,ds$

$\frac{3}{4}du = s^{1/3}\,ds$

$$\int s^{1/3}\cos(s^{4/3} - 8)\,ds = \frac{3}{4}\int \cos u\,du$$
$$= \frac{3}{4}\sin u + C$$
$$= \frac{3}{4}\sin(s^{4/3} - 8) + C$$

20. $\int \frac{dx}{\sin^2 3x} = \int \csc^2 3x\,dx$

Let $u = 3x$

$du = 3\,dx$

$\frac{1}{3}\,du = dx$

$$\int \csc^2 3x\,dx = \frac{1}{3}\int \csc^2 u\,du$$
$$= -\frac{1}{3}\cot u + C$$
$$= -\frac{1}{3}\cot(3x) + C$$

21. Let $u = \cos(2t + 1)$

$du = -\sin(2t + 1)(2)\,dt$

$-\frac{1}{2}du = \sin(2t + 1)\,dt$

$$\int \frac{\sin(2t + 1)}{\cos^2(2t + 1)}\,dt = -\frac{1}{2}\int u^{-2}\,du$$
$$= \frac{1}{2}u^{-1} + C$$
$$= \frac{1}{2\cos(2t + 1)} + C$$
$$= \frac{1}{2}\sec(2t + 1) + C$$

22. Let $u = 2 + \sin t$

$du = \cos t\,dt$

$$\int \frac{6\cos t}{(2 + \sin t)^2}\,dt = 6\int u^{-2}\,du$$
$$= -6u^{-1} + C$$
$$= -\frac{6}{2 + \sin t} + C$$

23. $\int_{\pi/4}^{3\pi/4} \cot x\,dx = \int_{\pi/4}^{3\pi/4} \frac{\cos x}{\sin x}\,dx$

Let $u = \sin x$

$du = \cos x\,dx$

$$\int_{\pi/4}^{3\pi/4} \frac{\cos x}{\sin x}\,dx = \int_{x = \pi/4}^{x = 3\pi/4} \frac{1}{u}\,du$$
$$= \ln|u|\Big]_{x = \pi/4}^{x = 3\pi/4}$$
$$= \ln|\sin x|\Big]_{\pi/4}^{3\pi/4}$$
$$= \ln\left|\frac{\sqrt{2}}{2}\right| - \ln\left|\frac{\sqrt{2}}{2}\right| = 0$$

24. Let $u = x + 2$

$du = dx$

$$\int_0^7 \frac{dx}{x + 2} = \int_2^9 \frac{1}{u}\,du$$
$$= \ln u\Big]_2^9$$
$$= \ln 9 - \ln 2 \approx 1.504$$

25. Let $u = x^2 + 1$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$\int_{-1}^{3} \frac{x\,dx}{x^2 + 1} = \frac{1}{2}\int_2^{10} \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u|\Big]_2^{10}$$
$$= \frac{1}{2}(\ln 10 - \ln 2) = \frac{1}{2}\ln 5 \approx 0.805$$

26. $\int_0^5 \frac{40\,dx}{x^2+25} = \int_0^5 \frac{\frac{40}{25}}{\left(\frac{x}{5}\right)^2 + \left(\frac{25}{25}\right)^2}\,dx$

$= \frac{40}{25}\int_0^5 \frac{1}{\left(\frac{x}{5}\right)^2 + 1}\,dx$

Let $u = \frac{x}{5}$

$du = \frac{1}{5}\,dx$

$5\,du = dx$

$\int_0^5 \frac{40\,dx}{x^2+25} = \frac{8}{5}(5)\int_0^1 \frac{1}{u^2+1}\,du$

$= 8\arctan u\Big]_0^1$

$= 8(\arctan 1)$

$= 8\left(\frac{\pi}{4}\right) = 2\pi$

27. $\int \frac{dx}{\cot 3x} = \int \frac{\sin 3x}{\cos 3x}\,dx$

Let $u = \cos 3x$

$du = -3\sin 3x\,dx$

$-\frac{1}{3}\,du = \sin 3x\,dx$

$\int \frac{dx}{\cot 3x} = -\frac{1}{3}\int \frac{1}{u}\,du$

$= -\frac{1}{3}\ln|u| + C$

$= -\frac{1}{3}\ln|\cos 3x| + C$

(An equivalent expression is $\frac{1}{3}\ln|\sec 3x| + C$.)

28. Let $u = 5x + 8$

$du = 5\,dx$

$\frac{1}{5}\,du = dx$

$\int \frac{dx}{\sqrt{5x+8}} = \frac{1}{5}\int u^{-1/2}\,du$

$= \frac{1}{5}\cdot 2u^{1/2} + C$

$= \frac{2}{5}\sqrt{5x+8} + C$

29. $\int \sec x\,dx = \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

Let $u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x\,dx$

$\int \sec x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sec x + \tan x| + C$

30. $\int \csc x\,dx = \int \csc x\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx$

$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$

Let $u = \csc x + \cot x$

$du = -\csc x \cot x - \csc^2 x\,dx$

$\int \csc x\,dx = -\int \frac{1}{u}\,du$

$= -\ln|u| + C$

$= -\ln|\csc x + \cot x| + C$

31. Let $u = y + 1$

$du = dy$

$\int_0^3 \sqrt{y+1}\,dy = \int_1^4 u^{1/2}\,du$

$= \frac{2}{3}u^{3/2}\Big]_1^4$

$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2}$

$= \frac{2}{3}(8) - \frac{2}{3} = \frac{14}{3}$

32. Let $u = 1 - r^2$

$du = -2r\,dr$

$-\frac{1}{2}\,du = r\,dr$

$\int_0^1 r\sqrt{1-r^2}\,dr = -\frac{1}{2}\int_1^0 u^{1/2}\,du$

$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_1^0$

$= -\frac{1}{3}(0) + \frac{1}{3}(1) = \frac{1}{3}$

33. Let $u = \tan x$

$du = \sec^2 x\,dx$

$\int_{-\pi/4}^0 \tan x \sec^2 x\,dx = \int_{-1}^0 u\,du$

$= \frac{1}{2}u^2\Big]_{-1}^0$

$= \frac{1}{2}(0) - \frac{1}{2}(-1)^2$

$= -\frac{1}{2}$

34. Let $u = 4 + r^2$

$du = 2r\,dr$

$\frac{1}{2}\,du = r\,dr$

$\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int_5^5 u^{-2}\,du = 0$

35. Let $u = 1 + \theta^{3/2}$

$du = \frac{3}{2}\theta^{1/2}\,d\theta$

$\frac{2}{3}\,du = \theta^{1/2}\,d\theta$

$$\int_0^1 \frac{10\sqrt{\theta}}{(1+\theta^{3/2})^2}\,d\theta = \frac{2}{3}(10)\int_1^2 u^{-2}\,du$$
$$= -\frac{20}{3}u^{-1}\Big]_1^2$$
$$= -\frac{20}{3}\left(\frac{1}{2} - 1\right)$$
$$= -\frac{20}{3}\left(-\frac{1}{2}\right) = \frac{10}{3}$$

36. Let $u = 4 + 3\sin x$

$du = 3\cos x\,dx$

$\frac{1}{3}\,du = \cos x\,dx$

$$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4 + 3\sin x}}\,dx = \frac{1}{3}\int_4^4 u^{-1/2}\,du = 0$$

37. Let $u = t^5 + 2t$

$du = (5t^4 + 2)\,dt$

$$\int_0^1 \sqrt{t^5 + 2t}\,(5t^4 + 2)\,dt = \int_0^3 u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2}\Big]_0^3$$
$$= \frac{2}{3}(3)^{3/2}$$
$$= \frac{2}{3}\sqrt{27} = 2\sqrt{3}$$

38. Let $u = \cos 2\theta$

$du = -2\sin 2\theta\,d\theta$

$-\frac{1}{2}\,du = \sin 2\theta\,d\theta$

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta = -\frac{1}{2}\int_1^{1/2} u^{-3}\,du$$
$$= -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)u^{-2}\Big]_1^{1/2}$$
$$= \frac{1}{4}\left(\left(\frac{1}{2}\right)^{-2} - 1\right)$$
$$= \frac{1}{4}(3) = \frac{3}{4}$$

39. $\frac{dy}{dx} = (y+5)(x+2)$

$\frac{dy}{y+5} = (x+2)dx$

Integrate both sides.

$\int \frac{dy}{y+5} = \int (x+2)\,dx$

On the left,

let $u = y + 5$

$du = dy$

$\int \frac{1}{u}\,du = \frac{1}{2}x^2 + 2x + C$

$\ln|u| = \frac{1}{2}x^2 + 2x + C$

$\ln|y+5| = \frac{1}{2}x^2 + 2x + C$

$|y+5| = e^{(1/2)x^2 + 2x + C}$

$|y+5| = e^C e^{(1/2)x^2 + 2x}$

We now let $C' = e^C$ or $C' = -e^C$, depending on whether

$(y + 5)$ is positive or negative. Then

$y + 5 = C'e^{(1/2)x^2 + 2x}$

$y = C'e^{(1/2)x^2 + 2x} - 5$

Since C' represents an arbitrary constant (note that even the

value $C' = 0$ gives a solution to the original differential

equation), we may write the solution as

$y = Ce^{(1/2)x^2 + 2x} - 5.$

40. $\frac{dy}{dx} = x\sqrt{y}\cos^2\sqrt{y}$

$\frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = x\,dx$

Integrate both sides.

$\int \frac{dy}{\sqrt{y}\cos^2\sqrt{y}} = \int x\,dx$

On the left, let $u = \sqrt{y}$

$du = \frac{1}{2}y^{-1/2}\,dy$

$2\,du = y^{-1/2}\,dy$

$2\int \frac{du}{\cos^2 u} = \frac{1}{2}x^2 + C$

$2\int \sec^2 u\,du = \frac{1}{2}x^2 + C$

$2\tan u = \frac{1}{2}x^2 + C$

$2\tan\sqrt{y} = \frac{1}{2}x^2 + C$

$\tan\sqrt{y} = \frac{1}{4}x^2 + C$

(Note: technically, C is now $C' = \frac{C}{2}$. But C's are generic.)

$\sqrt{y} = \tan^{-1}\left(\frac{x^2}{4} + C\right)$

$y = \left[\tan^{-1}\left(\frac{x^2}{4} + C\right)\right]^2$

41. $\frac{dy}{dx} = (\cos x)e^{y+\sin x}$

$\frac{dy}{dx} = (\cos x)(e^y e^{\sin x})$

$\frac{dy}{e^y} = \cos x\, e^{\sin x}\, dx$

Integrate both sides.

$\int \frac{dy}{e^y} = \int \cos x\, e^{\sin x}\, dx$

On the right, let $u = \sin x$

$du = \cos x\, dx$

$-e^{-y} = \int e^u\, du$

$-e^{-y} = e^u + C$

$-e^{-y} = e^{\sin x} + C$

$e^{-y} = -e^{\sin x} + C$

(Note: technically C is now $C' = -C$.)

$-y = \ln(C - e^{\sin x})$

$y = -\ln(C - e^{\sin x})$

42. $\frac{dy}{dx} = e^{x-y}$

$\frac{dy}{dx} = e^x e^{-y}$

$\frac{dy}{e^{-y}} = e^x\, dx$

Integrate both sides.

$\int \frac{dy}{e^{-y}} = \int e^x\, dx$

$\int e^y\, dy = \int e^x\, dx$

$e^y = e^x + C$

$y = \ln(e^x + C)$

43. $\frac{dy}{dx} = -2xy^2$

$-\frac{dy}{y^2} = 2x\, dx$

$-\int \frac{dy}{y^2} = \int 2x\, dx$

$y^{-1} = x^2 + C$

$y = \frac{1}{x^2 + C}$

$y(1) = \frac{1}{1 + C} = 0.25$

$1 + C = 4$

$C = 3$

$y = \frac{1}{x^2 + 3}$

44. $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}$

$\frac{dy}{\sqrt{y}} = 4\,\frac{\ln x}{x}\, dx$

Integrate both sides.

$\int \frac{dy}{\sqrt{y}} = 4\int \frac{\ln x}{x}\, dx$

On the right, let $u = \ln x$

$du = \frac{1}{x}\, dx$

$2y^{1/2} = 4\int u\, du$

$2y^{1/2} = 4\left(\frac{1}{2}u^2\right) + C$

$2y^{1/2} = 2(\ln x)^2 + C$

$y^{1/2} = (\ln x)^2 + C$

$y = [(\ln x)^2 + C]^2$

$y(e) = [(\ln e)^2 + C]^2 = 1$

$(1 + C)^2 = 1$

$C = 0$

$y = (\ln x)^4$

Note: Absolute value signs are not needed because the original problem involved $\ln x$, so we know that $x > 0$.

45. (a) Let $u = x + 1$

$du = dx$

$\int \sqrt{x+1}\, dx = \int u^{1/2}\, du$

$= \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}(x+1)^{3/2} + C$

Alternatively, $\frac{d}{dx}\left(\frac{2}{3}(x+1)^{3/2} + C\right) = \sqrt{x+1}$.

(b) By Part 1 of the Fundamental Theorem of Calculus, $\frac{dy_1}{dx} = \sqrt{x+1}$ and $\frac{dy_2}{dx} = \sqrt{x+1}$, so both are antiderivatives of $\sqrt{x+1}$.

(c) Using NINT to find the values of y_1 and y_2, we have:

x	0	1	2	3	4
y_1	0	1.219	2.797	4.667	6.787
y_2	−4.667	−3.448	−1.869	0	2.120
$y_1 - y_2$	4.667	4.667	4.667	4.667	4.667

$C = 4\frac{2}{3}$

(d) $C = y_1 - y_2$

$= \int_0^x \sqrt{x+1}\, dx - \int_3^x \sqrt{x+1}\, dx$

$= \int_0^x \sqrt{x+1}\, dx + \int_x^3 \sqrt{x+1}\, dx$

$= \int_0^3 \sqrt{x+1}\, dx$

46. **(a)** $\frac{d}{dx}[F(x) + C]$ should equal $f(x)$.

(b) The slope field should help you visualize the solution curve $y = F(x)$.

(c) The graphs of $y_1 = F(x)$ and $y_2 = \int_0^x f(t)\,dt$ should differ only by a vertical shift C.

(d) A table of values for $y_1 - y_2$ should show that $y_1 - y_2 = C$ for any value of x in the appropriate domain.

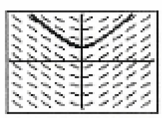
(e) The graph of f should be the same as the graph of NDER of $F(x)$.

(f) First, we need to find $F(x)$. Let $u = x^2 + 1$, $du = 2x\,dx$.

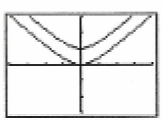
$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{2}u^{-1/2}\,du$$
$$= u^{1/2}$$
$$= \sqrt{x^2+1} + C$$

Therefore, we may let $F(x) = \sqrt{x^2+1}$.

a) $$\frac{d}{dx}(\sqrt{x^2+1} + C) = \frac{1}{2\sqrt{x^2+1}}(2x)$$
$$= \frac{x}{\sqrt{x^2+1}} = f(x)$$

b) 

[−4, 4] by [−3, 3]

c) 

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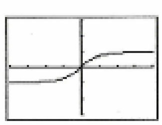
d)

x	0	1	2	3	4
y_1	1.000	1.414	2.236	3.162	4.123
y_2	0.000	0.414	1.236	2.162	3.123
$y_1 - y_2$	1	1	1	1	1

e) 

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47. Let $u = x^4 + 9$, $du = 4x^3\,dx$.

(a) $$\int_0^1 \frac{x^3\,dx}{\sqrt{x^4+9}} = \int_9^{10} \frac{1}{4}u^{-1/2}\,du = \frac{1}{2}u^{1/2}\Big]_9^{10}$$
$$= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9}$$
$$= \frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$$

(b) $$\int \frac{x^3}{\sqrt{x^4+9}}\,dx = \int \frac{1}{4}u^{-1/2}\,du$$
$$= \frac{1}{2}u^{1/2} + C$$
$$= \frac{1}{2}\sqrt{x^4+9} + C$$
$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \frac{1}{2}\sqrt{x^4+9}\Big]_0^1$$
$$= \frac{1}{2}\sqrt{10} - \frac{1}{2}\sqrt{9}$$
$$= \frac{1}{2}\sqrt{10} - \frac{3}{2} \approx 0.081$$

48. Let $u = 1 - \cos 3x$, $du = 3 \sin 3x\,dx$.

(a) $$\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx = \int_1^2 \frac{1}{3}u\,du = \frac{1}{6}u^2\Big]_1^2$$
$$= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

(b) $$\int (1 - \cos 3x)\sin 3x\,dx = \int \frac{1}{3}u\,du$$
$$= \frac{1}{6}u^2 + C$$
$$= \frac{1}{6}(1 - \cos 3x)^2 + C$$
$$\int_{\pi/6}^{\pi/3} (1 - \cos 3x)\sin 3x\,dx = \frac{1}{6}(1 - \cos 3x)^2\Big]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

49. We show that $f'(x) = \tan x$ and $f(3) = 5$, where

$$f(x) = \ln\left|\frac{\cos 3}{\cos x}\right| + 5.$$
$$f'(x) = \frac{d}{dx}\left(\ln\left|\frac{\cos 3}{\cos x}\right| + 5\right)$$
$$= \frac{d}{dx}(\ln|\cos 3| - \ln|\cos x| + 5)$$
$$= -\frac{d}{dx}\ln|\cos x|$$
$$= -\frac{1}{\cos x}(-\sin x) = \tan x$$
$$f(3) = \ln\left|\frac{\cos 3}{\cos 3}\right| + 5 = (\ln 1) + 5 = 5$$

50. (a) $u = \cot 2\theta$, $du = -2\csc^2 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, d\theta = -\frac{1}{2}\int u\, du = -\frac{1}{2}\cdot\frac{u^2}{2} + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$$

$$F_1(\theta) = -\frac{1}{4}\cot^2 2\theta$$

(b) $u = \csc 2\theta$, $du = -2\csc 2\theta \cot 2\theta\, d\theta$

$$\int \csc^2 2\theta \cot 2\theta\, du = -\frac{1}{2}\int u\, du = -\frac{1}{2}\cdot\frac{u^2}{2} + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$$

$$F_2(\theta) = -\frac{1}{4}\csc^2 2\theta$$

(c) $F_1'(\theta) = \left(-\frac{1}{2}\cot 2\theta\right)(-2\csc^2 2\theta) = \csc^2 2\theta \cot 2\theta$

$$F_2'(\theta) = \left(-\frac{1}{2}\csc 2\theta\right)(-2\csc 2\theta \cot 2\theta) = \csc^2 2\theta \cot 2\theta$$

(d) $F_1(\theta) = F_2(\theta) + b$

$$-\frac{1}{4}\cot^2 2\theta = -\frac{1}{4}\csc^2 2\theta + b$$

$$b = \frac{1}{4}(\csc^2 2\theta - \cot^2 2\theta) = \frac{1}{4}\left(\frac{1 - \cos^2 2\theta}{\sin^2 2\theta}\right) = \frac{1}{4}\left(\frac{\sin^2 2\theta}{\sin^2 2\theta}\right) = \frac{1}{4}$$

51. (a) $u = \sin x$, $du = \cos x\, dx$

$$\int 2\sin x \cos x\, dx = \int 2u\, du = u^2 + C = \sin^2 x + C$$

(b) $u = \cos x$, $du = -\sin x\, dx$

$$\int 2\sin x \cos x\, dx = \int (-2u)\, du = -u^2 + C = -\cos^2 x + C$$

(c) $u = 2x$, $du = 2\, dx$

$$\int 2\sin x \cos x\, dx = \int \sin 2x\, dx = \int \frac{1}{2}\sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos 2x + C$$

(d) $\frac{d}{dx}(\sin^2 x + C) = 2\sin x \cos x$

$$\frac{d}{dx}(-\cos^2 x + C) = (-2\cos x)(-\sin x) = 2\sin x \cos x$$

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x + C\right) = \left(\frac{1}{2}\sin 2x\right)(2) = \sin 2x = 2\sin x \cos x$$

■ Section 6.3 Integration by Parts (pp. 323–329)

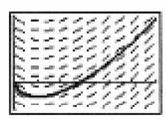
Exploration 1 Evaluating and Checking Integrals

1. $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = dx \Rightarrow v = x$. Thus,

$$\int \ln x\, dx = \int u\, dv = uv - \int v\, du = x\ln x - \int dx = x\ln x - x + C$$

2. $\frac{d}{dx}(x\ln x - x) = \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x$

3. The slope field of $\frac{dy}{dx} = \ln x$ shows the direction of the curve as it is graphed from left to right across the window.



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