Calculus
Chap. 5 section 5
Definite Integrals trapezoidal rule

Name: ______________________
Date: ______________________
Period: ______________________

1.

Use the trapezoidal rule with $n = 4$ to approximate the value of the integral $\int_{5}^{11} 2x\,dx$. Then use the concavity of the function to predict whether the approximation is an overestimate or an underestimate, and check your answer by finding the exact value of the integral.

The trapezoidal rule approximates the integral $\int_{a}^{b} f(x)dx$ by partitioning the area under the curve $y = f(x)$ into subintervals containing narrow trapezoids of equal width whose vertical bases have lengths equal to the values of $f(x)$ at the endpoints of each subinterval.

1. answer

Use the trapezoidal rule with $n = 4$ to approximate the value of the integral $\int_5^{11} 2x\,dx$. Then use the concavity of the function to predict whether the approximation is an overestimate or an underestimate, and check your answer by finding the exact value of the integral.

The trapezoidal rule approximates the integral $\int_a^b f(x)dx$ by partitioning the area under the curve $y = f(x)$ into subintervals containing narrow trapezoids of equal width whose vertical bases have lengths equal to the values of f(x) at the endpoints of each subinterval.

The total area, T, of these trapezoids is shown below.

$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$ where [a,b] is partitioned into n equal subintervals of length $h = \frac{(b-a)}{n}$

For $\int_5^{11} 2x\,dx$, h is $\frac{3}{2}$.

For $x_0 = 5$ and $x_4 = 11$, $n = 4$ gives $x_1 = 6.5$, $x_2 = 8$, and $x_3 = 9.5$.

Evaluate f(x) at each of these values of x.

x	y = f(x)
5	2(5) = 10
6.5	2(6.5) = 13
8	2(8) = 16
9.5	2(9.5) = 19
11	2(11) = 22

Substitute the values for h and y_0, y_1, y_2, y_3, and y_4 into T and evaluate.

$$T = \frac{1.5}{2}(10 + 2 \cdot 13 + 2 \cdot 16 + 2 \cdot 19 + 22)$$

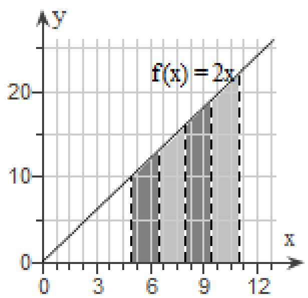
$$= 96$$

The trapezoidal rule approximation of $\int_5^{11} 2x\,dx$ for $n = 4$ is 96.

To determine whether this approximation is an overestimate or an underestimate, consider the graph of the integrand below.

1. answer cont.

The graph of f(x) in the interval [5,11] is neither concave up nor concave down.



Because f(x) is neither concave up nor concave down in the interval [5,11], the trapezoid rule approximation is neither an overestimate nor an underestimate. In other words, it results in an exact value for $\int_5^{11} 2x\,dx$.

To verify this, use the integral evaluation theorem below to find an exact value for $\int_5^{11} 2x\,dx$ on the interval [5,11].

$\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f on [a,b]

First, apply the constant multiple rule for definite integrals to the integral.

$$\int_5^{11} 2x\,dx = 2\int_5^{11} x dx$$

Now, find an antiderivative of f(x).

$$= 2\left[\frac{x^2}{2}\right]_5^{11}$$

Evaluate $F(b) - F(a)$.

$$= 2\left(\frac{11^2}{2} - \frac{5^2}{2}\right)$$

$$= 96$$

Therefore, the exact value of $\int_5^{11} 2x\,dx$ is 96, which is equal to the trapezoid rule approximation T found previously.

Calculus
Chap. 5 section 5
Definite integrals & trapezoidal rule

Name: ______________________
Date: ______________________
Period: ______________________

2.

Use the trapezoidal rule with $n = 4$ to approximate the value of the integral $\int_6^{18} \frac{2}{x}\,dx$. Then use the concavity of the function to predict whether the approximation is an overestimate or an underestimate, and check your answer by finding the exact value of the integral.

The trapezoidal rule approximates the integral $\int_a^b f(x)\,dx$ by partitioning the area under the curve $y = f(x)$ into subintervals containing narrow trapezoids of equal width whose vertical bases have lengths equal to the values of $f(x)$ at the endpoints of each subinterval.

2. answer

Use the trapezoidal rule with $n = 4$ to approximate the value of the integral $\int_6^{18} \frac{2}{x}\,dx$. Then use the concavity of the function to predict whether the approximation is an overestimate or an underestimate, and check your answer by finding the exact value of the integral.

The trapezoidal rule approximates the integral $\int_a^b f(x)\,dx$ by partitioning the area under the curve $y = f(x)$ into subintervals containing narrow trapezoids of equal width whose vertical bases have lengths equal to the values of f(x) at the endpoints of each subinterval.

The total area, T, of these trapezoids is shown below.

$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$ where [a,b] is partitioned into n equal subintervals of length $h = \frac{(b-a)}{n}$

For $\int_6^{18} \frac{2}{x}\,dx$, h is 3.

For $x_0 = 6$ and $x_4 = 18$, $n = 4$ gives $x_1 = 9$, $x_2 = 12$, and $x_3 = 15$.

Find y at each of these values of x by evaluating $\frac{2}{x}$.

x	y = f(x)
6	$\frac{1}{3}$
9	$\frac{2}{9}$
12	$\frac{1}{6}$
15	$\frac{2}{15}$
18	$\frac{1}{9}$

Substitute the values for h and y_0, y_1, y_2, y_3, and y_4 into T and evaluate.

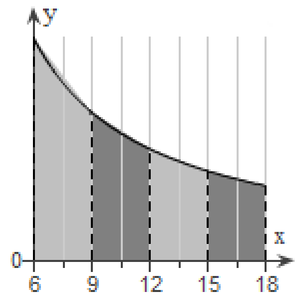
$$T = \frac{(3)}{2}\left(\frac{1}{3} + 2 \cdot \frac{2}{9} + 2 \cdot \frac{1}{6} + 2 \cdot \frac{2}{15} + \frac{1}{9}\right)$$

2. answer cont.

The trapezoidal rule approximation of $\int_6^{18} \frac{2}{x}\,dx$ for $n = 4$ is 2.233 (rounded to the nearest thousandth).

To determine whether this approximation is an overestimate or an underestimate, consider the graph of the integrand below.

The graph of $\frac{2}{x}$ is concave up in the interval [6,18].



Because f(x) is concave up in the interval [6,18], the approximating segments lie above the curve, making the trapezoid rule approximation an overestimate of the value of $\int_6^{18} \frac{2}{x}\,dx$.

To verify this, use the integral evaluation theorem below to find an exact value for $\int_6^{18} \frac{2}{x}\,dx$ on the interval [6,18].

$\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f on [a,b]

First, apply the constant multiple rule for definite integrals to the integral.

$$\int_6^{18} \frac{2}{x}\,dx = 2\int_6^{18} \frac{1}{x}dx$$

Now, find an antiderivative of $\frac{1}{x}$.

$$= 2\,[\mathbf{ln}\ x]_6^{18}$$

Evaluate $F(b) - F(a)$.

$$= 2(\mathbf{ln}\ 18 - \mathbf{ln}\ 6)$$

$$\approx 2 \cdot 1.099$$

Simplify.

$$\approx 2.198$$

Therefore, the exact value of $\int_6^{18} \frac{2}{x}\,dx$ is about 2.198, which is less than the trapezoid rule approximation, $T \approx 2.233$, found previously.

Calculus
Chap. 5 section 5
Definite Integrals & trapezoidal rule

Name: ______________________________
Date: ______________________________
Period: _____________________________

3. Use the function values in the following table and the trapezoidal rule with $n = 6$ to approximate $\int_4^{10} f(x)\,dx$.

x	4	5	6	7	8	9	10
f(x)	10	8	7	11	14	17	18

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdot\cdot\cdot + 2y_{n-1} + y_n),$$

where $[a,b]$ is partitioned into n subintervals of equal length $h = (b - a) / n$.

3. answer

Use the function values in the following table and the trapezoidal rule with $n = 6$ to approximate $\int_4^{10} f(x)\,dx$.

x	4	5	6	7	8	9	10
f(x)	10	8	7	11	14	17	18

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdot\cdot\cdot + 2y_{n-1} + y_n),$$

where $[a,b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.

To use the trapezoidal rule first determine the height of each trapezoid.

$$h = \frac{10 - 4}{6}$$
$$= 1$$

Substitute the correct values into the trapezoid equation and simplify.

$$T = \frac{1}{2}(10 + 2(8) + 2(7) + 2(11) + 2(14) + 2(17) + 18)$$
$$= \frac{1}{2}(142)$$
$$= 71$$

Thus, $\int_4^{10} f(x)\,dx \approx 71$.

Calculus Chap. 5 section 5 Definite Integrals & trapezoidal rule	Name: ____________________ Date: ____________________ Period: ____________________

4.

Use Simpson's Rule with $n = 4$ to approximate the value of the integral $\int_1^{13} 10x\,dx$. Then check your answer by finding the exact value of the integral.

Simpson's Rule approximates the integral $\int_a^b f(x)dx$ by partitioning the area under the curve $y = f(x)$ into an even number of subintervals and constructing a parabolic arc that passes through the values of $f(x)$ at the endpoints of every other subinterval.

4. answer

Use Simpson's Rule with $n=4$ to approximate the value of the integral $\int_1^{13} 10x\,dx$. Then check your answer by finding the exact value of the integral.

Simpson's Rule approximates the integral $\int_a^b f(x)dx$ by partitioning the area under the curve $y=f(x)$ into an even number of subintervals and constructing a parabolic arc that passes through the values of $f(x)$ at the endpoints of every other subinterval.

The total area, S, under these parabolic arcs is shown below.

$S=\frac{h}{3}(y_0+4y_1+2y_2+4y_3+\ldots+2y_{n-2}+4y_{n-1}+y_n)$ where $[a,b]$ is partitioned into an even number n of equal subintervals of length $h=\frac{(b-a)}{n}$

For $\int_1^{13} 10x\,dx$, h is 3.

For $x_0=1$ and $x_4=13$, $n=4$ gives $x_1=4$, $x_2=7$, and $x_3=10$.

Find y at each of these values of x by evaluating $f(x)=10x$.

x	y = f(x)
1	10
4	40
7	70
10	100
13	130

Substitute the values for h and y_0, y_1, y_2, y_3, and y_4 into S and evaluate.

$$S=\frac{3}{3}(10+4\cdot 40+2\cdot 70+4\cdot 100+130)$$

The Simpson's Rule approximation of $\int_1^{13} 10x\,dx$ for $n=4$ is 840.

To check this approximation, use the integral evaluation theorem below to find an exact value for $\int_1^{13} 10x\,dx$ on the interval [1,13].

$\int_a^b f(x)dx=F(b)-F(a)$ where F is any antiderivative of f on [a,b]

4. answer cont.

First, apply the constant multiple rule for definite integrals to the integral.

$$\int_{1}^{13} 10x\,dx = 10\int_{1}^{13} x\,dx$$

Now, find an antiderivative of f(x).

$$= 10\left[\frac{x^2}{2}\right]_{1}^{13}$$

Evaluate $F(b) - F(a)$.

$$= 10 \cdot \left(\frac{13^2}{2} - \frac{1^2}{2}\right)$$

$$= 10 \cdot 84$$

$$= 840$$

Therefore, the exact value of $\int_{1}^{13} 10x\,dx$ is 840, which is equal to the Simpson's Rule approximation S found previously.

Calculus
Chap. 5 section 5
Definite Integrals & trapezoidal rule

Name: ______________________
Date: ______________________
Period: ______________________

5. Use a calculator program to find the Simpson's Rule approximations with $n = 50$ and $n = 100$.

$$\int_0^{\pi/6} \frac{\sin x}{x}\,dx$$

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdot\cdot\cdot + 2y_{n-2} + 4y_{n-1} + y_n),$$

where $[a,b]$ is partitioned into an even number n of subintervals of equal length $h = (b - a) / n$.

5. answer

Use a calculator program to find the Simpson's Rule approximations with $n = 50$ and $n = 100$.

$$\int_0^{\pi/6} \frac{\sin x}{x}\, dx$$

Simpson's Rule

To approximate $\int_a^b f(x)\, dx$, use

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n),$$

where $[a,b]$ is partitioned into an even number n of subintervals of equal length $h = (b - a) / n$.

To use Simpson's Rule start by finding the length of the subintervals. First calculate h when $n = 50$.

$$h = \frac{(\pi / 6) - 0}{50}$$

$$= \frac{\pi}{300}$$

Now use a calculator program to approximate the integral using Simpson's Rule. Since the function is undefined at $x = 0$, use an initial value of 0.0001 for your calculations.

$S_{50} = 0.5156$

Calculate h when $n = 100$.

$$h = \frac{(\pi / 6) - 0}{100}$$

$$= \frac{\pi}{600}$$

Now use a calculator program to approximate the integral using Simpson's Rule. Since the function is undefined at $x = 0$, use an initial value of 0.0001 for your calculations.

$S_{100} = 0.5156$

Thus, $\int_0^{\pi/6} \frac{\sin x}{x}\, dx \approx 0.5156$. This answer is accurate to four decimal places because S_{50} and S_{100} are the same when calculated to four places.