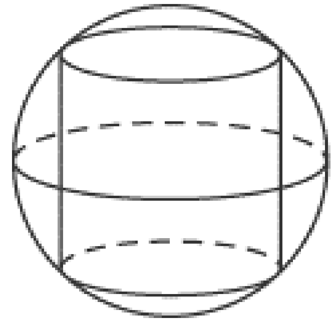
CALCULUS NOTES: FINDING EXTREMA

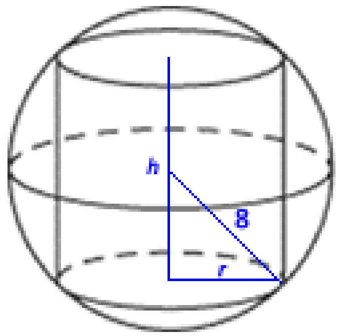


Problem #1: Joey wants to chill pop cans in a spherical "ice shell" made of a biodegradable gel. As part of his research, he his thinking that he should create pop cans perhaps in a different size than typical. Hence he wants to find the dimensions of a right cylinder with maximal volume that can be inscribed in a sphere of radius 8cm.

Sample Solution:

STEP 1. DRAW A PICTURE TO INCLUDE DETAILS AND IMPORTANT VARIABLES.

Let h = cylinder height
Let r = cylinder radius



STEP 2. WRITE A FUNCTION OF THE "THING" YOU'RE TRYING TO OPTIMIZE IN TERMS OF A SINGLE VARIABLE.

Optimizing volume… $V = \pi r^2 h$
Is there an equation relating r and h? If so, I can solve for r or h and get volume in terms of one variable.

I see a right triangle. So $r^2 + \left(\frac{h}{2}\right)^2 = 8^2$.

That means $r^2 = 64 - \frac{h^2}{4}$. Hey, I have an r^2 in my volume formula!

So my equation $V = \pi r^2 h$ becomes $V = \pi\left(64 - \frac{h^2}{4}\right)h$

My volume function: $V(h) = \pi\left(64 - \frac{h^2}{4}\right)h$

STEP 3. FIND CRITICAL POINTS AND DOMAIN RESTRICTIONS (ENDPOINTS).

$V(h) = \pi\left(64 - \frac{h^2}{4}\right)h$ can be rewritten as $V(h) = 64\pi h - \frac{\pi}{4}h^3$.

$V'(h) = 64\pi - \frac{3\pi}{4}h^2$

$V'(h) = 0 \quad \Rightarrow \quad h = \pm 9.238$

$V'(h)$ DNE never.

Domain: $0 < h < 16$

Critical Points and Endpoints: h = 0, 9.238, 16

STEP 4. TEST THE CRITICAL POINTS AND ENDPOINTS.

Option 1: Brute Force Method

Critical Points and Endpoints: h = 0, 9.238, 16

$V(0)=0$

$V(9.238)=1238.220$

$V(16)=0$

There cannot be any other higher or lower function values than those.
Therefore the maximum volume is 1238.220 cm^3 when h = 9.238 cm.

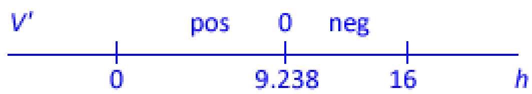
Option 2: First Derivative Test

Test the values of $V'(h)$ around the critical points and endpoints:

$V'(2)>0$

$V'(12)<0$

I then have the following sign pattern chart:



V is increasing immediately after h = 0, so $V(0)$ is a local minimum.
V is decreasing immediately before h = 16, so $V(16)$ is a local minimum.

Therefore, $V(9.238)=1238.220$ is an absolute maximum since V' changes sign from positive to negative at h = 9.238. (That is, V changes from increasing to decreasing at h = 9.238.)

Thus the maximum volume is 1238.220 cm^3 when h = 9.238 cm.

Option 3: Second Derivative Test

Test the values of $V''(h)$ when $V'(h)=0$.

$$V''(h)=\frac{-3\pi}{2}h$$

$V''(9.238)<0$

Since $V''(9.238)<0$ and $V'(9.238)=0$, the maximum is $V(9.238)=1238.220$.
(This means that the slopes were decreasing, became zero, and then continue to decrease.)

Thus the maximum volume is 1238.220 cm^3 when h = 9.238 cm.

STEP 5. VERIFY ON A GRAPH, IF POSSIBLE.